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A Short Description on Locally Convex and Polar Topologies

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ABSTRACT: *In this paper an attempt has been made to make some positive contributions to the study of topological dynamical systems. It is well known that the classical studies of dynamical systems are based on Hamilton's principle and the principle of least action and techniques employed in them is usually the variational calculus. Some years ago, studies has been made of dynamical systems which carry topologies on them making the dynamical behavior of the system continuous i.e. the so-called topologies campactible with dynamical structure of the system. This relates the study of dynamical systems with the study of continuous semigroup of operators on Banach Algebras, specially C^*-Algebra, and W^*-Algebras. The most useful topologies for such studies are so-called weak topologies, strong topologies and the corresponding notations of weak continuities and strong continuities. Thus we are lead to the studies of topological dynamical systems in the context of topological vector space, specially locally convex spaces and locally compact groups.*

KEYWORDS - Absorbent, Absolutely Convex, Balanced, Convex, , Locally Convex Topological Linear Space,

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I. INTRODUCTION

It is well known that for a conservative dynamical system whose state can be described by n generalized co-ordinates q_1, q_2, q_3, q_4, q_5,…………, q_n, at time t , the kinetic energy T of the system is a quadratic function of the generalized components of the velocities q_1^*, q_2^*, q_3^*, q_4^*, q_5^*, ………., q_n^* (and when time t is not explicitly involved), T is a homogeneous quadratic function of the generalized velocities. The potential energy V of the system is a function of q_1, q_2, q_3, q_4, q_5,…………, q_n and t. The notion of such a system is determined by Hamilton's principle which states dynamical system that " the motion of a conservative dynamical system between times t_1 and t_2 proceeds in such a way that the functional $J = \int_{t_1}^{t_2}(T - V)dt$ is stationary".

Writing L = T – V, the well known Euler-Lagrange equation $\frac{d}{dt}(\frac{\partial L}{\partial q_k^*} - \frac{\partial L}{\partial q_k}) = 0$, k = 1,2,3,……..,n are obtained.

The second well known principle for a dynamical system observing law of conservation of energy T + V = C (Constant), Where T is independent of time (and consequently, L=T-V=2T-C, is also independent of time) is the "principle of least action" which states that "if the time t is not an independent variable but the notion is dependent upon other independent variable r which is to assume fixed values at the end points, the action A of the dynamical system give by $A = \int_{r_1}^{r_2}(2T \frac{dt}{dr})dr$ (where $r_1$and r_2 are fixed) is stationary. It can be shown to be a minimum.

In addition classical mathematical physics contains a large number of well known results and techniques concerning vibration and eigen value problems, special functions defined by them and the technique of calculus of variations for solving some such problems. The literature dealing with classical methods is extensive. The classical problems are quadratic, the related Euler-Langrage equations linear and what is called nowadays formally self adjoint.

The main aim of the research paper is to use topological and functional analytic technique for solving classes of such mechanical problems instead of individual problems. A reasonable frame work for such a study is a Hilbert space H and the operators on H. Some of the result can then be generalized to Banach spaces and even to locally convex topological vector spaces and the theory of modern operator algebras associated with these space, while operator algebra of bounded operators suffice for a large class of problems, for solving problems in quantum mechanics, the theory of unbounded operators from an essential ingredient . In fact, some of the most useful operators defined in a Hilbert space are the differential operators which are rarely defined on the whole space and which are almost never bounded. Such operators occur in quantum mechanics. In classical mechanics, the total energy of the system under consideration is given in general, by its Hamiltonian function.

H=H(q,p), which depends on the position co-ordinate q and the momentum co-ordinate p of the particle. In the quantum mechanical description, we replace q and p by the position operator Q and the momentum operator P respectively, and obtain the Hamiltonian operator H=H(Q,P), It is well known that Q and P do not commute with each other i.e. QP - PQ $\neq$ 0. The equation H(Ψ)(s)=μΨ(s), where Ψ ϵ D(H), (Domain of H) and μ is a scalar, is known as the time –independent Schrodinger equation. Also it is known as "a pure state". of the system, sometimes it is also known as a wave function.

An eigen value of the Hamiltonian operator H i.e. a scalar μ for which the above equation has a non-zero solution Ψ ϵ D(H), represents a quantized energy level of the system under consideration.

II. DEFINITIONS, RESULTS AND PROOFS

In this section firstly I shall give definitions and then established some of results as in the form of proposition and then the results will be prove with reference of the above definitions, notions and preliminaries.

1.0 LOCALLY CONVEX TOPOLGIES

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1.1 Def. - A vector space X over a field K is called **topological vector space** or a topological linear space X(J) if a topology J in defined on X which is compatible with the vector space structure in the sense that the mapping (x,y)→x+y from X × X into X and (α,X)→αX from K×X into X are continuous.
The above concept was defined by J.Von Neumann.

A normed linear space is an example of topological linear space giving a norm, however, provides a richer structure than that provided by the topology which it induces, since two normed spaces can be topologically isomorphic as topological vector spaces without being norm isomorphic.

1.2 Prop. -Given a topological vector space X(J) the mapping $X \to X + X_0$ is a homomorphic of X(J) onto itself and the mapping X→αX, α≠0, is a topological automorphism of X.
Proof :-In fact, the mapping $\mathbf{X \to X + X_0}$ is continuous, one-one and has the whole of X as image space. The inverse mapping $\mathbf{X \to X - X_0}$ is also continuous. Thus $\mathbf{X \to X + X_0}$ is a homomorphism.
The mapping **X→αX, α≠0** is also one-one linear and continuous and the whole of X as image space. The inverse mapping $\mathbf{X \to \frac{1}{\alpha}X, \alpha \neq 0}$ exists and has the same properties.Thus **X→αX, α≠0** is a topological automorphism of X.

1.3 Remark - In view of above proposition, if u = {U} is a fundamental system of neighborhoods of origin in a topological vector space X(J) then $\{x_0+U : U \in u\}$ form a base for the neighborhood filter of $x_0 \in X$. Moreover, if U is a neighborhood of origin in X, so is αU with α≠0.

1.4 Def.– Given a topological vector space X(J), a subset M of X is said to be **absorbent** if given any x ϵ X, there exists $\rho > 0$ such that ρx ϵ M. M is said to be **balanced** if whenever |α| ≤ 1 and x_0 ϵ M then αx_0 ϵM .

1.5 Prop-If X(J) be a topological vector space over a field K and u = {U} is a fundamental system of neighborhoods of origin in X then (i) For each U ϵ u there exists V ϵ u with V + V ⊆ U.(ii) For each U ϵ u there exists V ϵ u for which αV ⊆ U for all α with |α| ≤ 1.(iii) Such U ϵ u is absorbent.
Proof- (i) Follow from the continuity of (x,y)→x+y at (0,0). It follows from the continuity of αx at (0,0) that there is an $\varepsilon > 0$ and a neighborhood W of origin in X such that x ϵ U for all x ϵ W and all $|\lambda| \leq \varepsilon$.
Hence (ii) is satisfied taking V = εW.
If U were not absorbent, there would be an x_0 ϵ X which would not be in any n . U for any positive integer n. Thus $\frac{1}{n} x_0 \notin U$ which contradicts the convergence of $\frac{1}{n} x_0$ to zero.

1.6 Corollary.- A topological vector space X has a fundamental system of neighborhood of origin consisting of balanced neighborhoods. In fact, from (ii) of Prop1.2. the set {αU: |α| ≤ 1, U ϵ u } from a fundamental system of neighborhoods of origin. It is also well known that given a filter base u = {U} on a vector space X satisfying conditions (i),(ii) and (iii) of prop.1.2., If a topology J is defined on X by taking as neighborhood of x ϵ X the sets U(x) = x + U, Uϵ u, then X(J) is a topological vector space with u as a fundamental system of neighborhood of origin.

1.7 Def.-A subset M of a linear space X is called **convex** if for all x,y ϵ M, λx + μy ϵ M whenever $\lambda \geq 0, \mu \geq 0$ and $\lambda + \mu$ =1.
M is said to be **absolutely convex** if it is both balanced and convex . This is equivalent to saying that for all x,y ϵ M, λx + μy ϵ M whenever $|\lambda| + |\mu| \leq 1$

1.8 Def. – A topological vector space X(J) is said to be a Locally convex topological vector space or simply convex space if it is fundamental system of convex neighborhood of the origin.

1.9 Prop.- A locally convex space X(J) has a fundamental system a of neighborhoods of origin satisfying the following properties.-

(c_1) If U ϵ u, V ϵ u, then there exists W ϵ u with W $\subseteq$ U $\cap$ V ;

(c_2) If U ϵ u, and $\alpha \neq 0$ then αU ϵ u ;

(c_3) Each U ϵ u is absolutely convex and absorbent.

Proof- If X is a locally convex space then by definition, X has a fundamental system of convex neighborhoods of origin. If U is one of them then $\cap\{\lambda U:|\lambda| \geq 1\}$is a balanced neighborhood of origin contained in U. It is also convex, since it is an intersection of convex sets. Thus there is a fundamental system v of absolutely convex neighborhoods of origin. Taking u = $\{\alpha V : \alpha \neq 0, V \epsilon v\}$, we find that u is a fundamental system of neighborhoods of origin satisfying (c_1),(c_2) and (c_3).

Remark – It is also well known that given a nonempty family u of subsets of a vector space X with the properties (c_1),(c_2) and (c_3) there is a topology making X a convex space with u as a fundamental system of neighborhoods of origin. The converse also holds.

2.0 THE POLAR TOPOLOGIES

Let X be a Banach space and let X^* be the Topological dual space of X, Then X^* is the Banach space of all continuous linear functional on X. Let F be a closed subspace of X^* and let σ(X,F) be the locally convex topology on X determined by F in the following sense :

Given $f_1, f_2, f_3, \ldots\ldots, f_n$ ϵ F, a seminorm p is defined on X by setting p(x) = $\sup_{1 \leq k \leq n} |f_k(x)|$ for x ∈ X.

The locally convex topology on X determined by the family of all such semi-norm (for various choices of f_k's in F) is denoted by σ(X,F) and is called the weak topology on X determined by F. A fundamental system of neighborhoods of origin in X for this topology is given by sets of the form W = {x ∈ X : p(x) ≤ ε} for $\varepsilon > 0$.

A subset of X is said to be σ(X,F) compact on weakly compact if it is a compact subset of X when X has weak topology determined by F. A subset of X is said to be σ(X,F) – closed or weakly closed if it is a closed subset of X when X has weak topology.

Given $x_1, x_2, x_3, \ldots\ldots x_n$ ∈ X, a seminorm q can be defined on F by setting, q(f) = $\sup_{1 \leq k \leq n} |f(x_k)|$ for f ∈ F.

The locally convex topology on F determined by the family of all such semi norms (for various choice of x_k's in X) is denoted by σ(X,F) and is called the **weak topology** on F determined by X. A fundamental system of neighborhoods of origin in F for this topology is given by sets of the form {f ∈ F : q(f) ≤ ε} for $\varepsilon > 0$.

The locally convex topologies defined above are special cases of the so called 'polar topologies' described below.

2.1 Def. Let X and Y be two vector spaces over the same field K, paired with respect to the bilinear form B(x,y) in the sense that B:X×X→K is a bilinear mapping. If M⊆X, then the polar of M is the subset of M^0 of Y defined by M^0 = {y ∈ Y: |B(x,y)| ≤ 1 for all x ∈ M}.

We first note that M^0 is a balanced, convex set in Y closed for σ(Y,X). In order to prove this result let y,z ∈ M^0, λ,μ ∈ K such that $|\lambda| + |\mu| \leq 1$. Then for any x ∈ M we have,

$$|B(x, \lambda y+ \mu z| \leq |\lambda| . |B(x,y)| +|\mu| . |B(x,z)| \leq |\lambda|+|\mu| \leq 1,$$

That is λy + μ z ∈ M^0 and hence M^0 is balanced and convex.

Again for each x ∈ X, the map y → B(x,y) is continuous on Y for σ(Y,X).

Hence the set M_x = {y ∈ Y: |B(x,y)| ≤ 1} is closed in Y on the inverse image of the closed set {α : |α| < 1}is closed in K. Thus $M^0 = \cap\{M_x : x \in M\}$ is closed for σ(Y,X).

Thus M^0 is balanced, convex set in Y closed for σ(Y,X).

Also M^0 is absorbent if and only if M is bounded for σ(X,Y).

In fact, if M^0 is absorbent and y ∈ X, there exists $\mu > 0$ such that μ_y ∈ M^0.

Then |B(μx,y) = |B(x, μy| ≤ 1 for all x ∈ M. Hence y ∈ $(\mu M)^0$. Thus $P_y(x)$ = |B(x,y)| ≤ $\frac{1}{\mu}$ for all x ∈ M. i.e. M is bounded for σ(X,Y). On the other hand, if M is bounded for σ(X,Y), then for each y ∈Y there exists $\mu > 0$ such that |B(x,y)| ≤ $\frac{1}{\mu}$ for all x ∈ M. Hence μy ∈ M^0 and thus M^0 is absorbent.

2.2 Prop – Let $(M_i)_{i \in I}$ be a collection of balanced convex non empty subsets of X which are closed for the topology σ(X,Y). Then the polar of $\cap_{i \in I} M_i$ is the balanced, convex σ(X,Y) – closed envelope of the union of the sets M_1^0.

Proof – By the theorem of bipolar we have $M_1 = M_1^{00}$.

Hence $\cap_{i \in I} M_i = \cap_{i \in I} M_i^{00} = (\cup_{i \in I} M_i^0)^0$.

Therefore $(\bigcap_{i \in I} M_i)^0 = (\bigcup_{i \in I} M_i^0)^{00}$.
But by the theorem of bipolar, the Right hand side is balanced, convex, σ(X,Y) – closed envelope of $\bigcup_{i \in I} M_i^0$.
This complete the proof.

2.3 Def – (Polar topology) Let S be a family of σ(X,Y) – bounded subsets of X. Then the polar M^0 of the sets M ∈ S form a family of absorbing, balanced, convex sets in Y and thus define a locally convex topology on Y, called the S – topology of uniform convergence on sets belonging to S or the polar topology on Y.

2.4 Prop – Let S be a family of balanced, convex, absorbing sets in a vector space Y. Then there is a weakest topology on Y compatible with the algebraic structure in which every set in S is a neighborhood of origin. Under this topology Y is a locally convex spaces and a base of neighborhood of origin is formed by the sets $\varepsilon \bigcap_{1 \leq i \leq n} V_i$, {$\varepsilon$ > 0,V_1 ∈ S}.

Proof – Consider the family γ of subsets of the form $\varepsilon \bigcap_{1 \leq i \leq n} V_i$, {$\varepsilon$ > 0,V_1 ∈ S}.
Since intersection of a finite number of balanced convex, absorbent sets is balanced, convex and absorbent, it follows that each member of γ is balanced, convex and absorbent. Also the scalar multiple by a non zero of each member of γ is in γ and intersection of any two numbers of γ contains a member of γ. Hence there is a topology J on Y making Y a convex space with γ as a fundamental system of neighborhoods of origin. Alas in any compatible topology on Y in which the sets of S are neighborhood of the origin, the sets of γ must also be a neighborhood since the intersection of two sets neighborhoods is neighborhood and scalar multiple of a neighborhood is a neighborhood. Thus J is the weakest such convex topology.

2.5 Remark – If S be a family of σ(X,Y) – balanced subsets of X, then the finite intersection of the sets λM^0, where λ > 0 and M ∈ S, form a fundamental J system of neighborhoods of origin for the S – topology on Y. The S – topology on Y can also be defined by the family of semi norm q_M where M ∈ S, which are given by q_M(y) = sup { |B(x,y)| : x ∈ M } since $q_M(y) \leq \varepsilon \Leftrightarrow y \in \varepsilon M^0$.

III. CONCLUSION

The main advantage of this paper is it expand the study of topological dynamical system and It can be say that a little effort have been done to link with applied mathematics and pure mathematics. The most important result of this paper is given with the help of the definition "A topological vector space X(J) is said to be a Locally convex topological vector space or simply convex space if it is fundamental system of convex neighborhood of the origin" is as follows-

A locally convex space X(J) has a fundamental system a of neighborhoods of origin satisfying the following properties.-
(c_1) If U ϵ u, V ϵ u, then there exists W ϵ u with W ⊆ U ⋂ V ;
(c_2) If U ϵ u, and α≠0 then αU ϵ u ;
(c_3) Each U ϵ u is absolutely convex and absorbent.
This will be very fruitful in further study of locally convex topological vector spaces.

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