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Concerning Non-Negative Quaternion Doubly Stochastic Matrices

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***Abstract**: This paper is concerned with the condition for the convergence to a quaternion doubly stochastic limit of a sequence of matrices obtained from a non-negative matrix A by alternately scaling the rows and columns of A and with the condition for the existence of diagonal matrices D_1 and D_2 with positive main diagonals such that $D_1 A D_2$ is quaternion doubly stochastic.The result is the following the sequence of matrices converges to a doubly stochastic limit if and only if the quaternion matrix A contains at least one positive main diagonal.A necessary and sufficient condition that there exists diagonal matrices D_1 and D_2 with positive main diagonal matrices such that $D_1 A D_2$ is both quaternion doubly stochastic and the limit of the iteration is that $A \neq 0$ and each positive entry of A is contained in a positive diagonal. The form $D_1 A D_2$ is unique, and D_1 and D_2 are unique up to a positive scalar multiple if and only if A is fully indecomposable.*

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I. Definitions:

If $A \in H^{n \times n}$ is a quaternion doubly stochastic matrix and σ is a Permutation of $\{1 \ldots \ldots n\}$, then the sequence of elements $a_1, \sigma_{(1) \ldots\ldots} a_n, \sigma_{(n)}$ is called the diagonal of A corresponding to σ. If σ is the identity, the diagonal is called the main diagonal.

If A is a non-negative square matrix, A is said to have total support if $A \neq o$ and if every positive element of A lies or a positive diagonal.

II. Theorem:

Let A be a nonnegative n x n quaternion matrix. A necessary and sufficient condition that there exists a quaternion doubly stochastic matrix B of the form $D_1 A D_2$ where D_1 and D_2 are diagonal matrices with positive main diagonals is that A has total support. If B exists then it is unique. Also D_1 and D_2 are unique upto a scalar multiple if and only if A is fully indecomposable.

A necessary and sufficient condition that the iterative process of alternately normalizing the rows and columns of A will converge to a quaternion doubly stochastic limit is that A has support. If A has total support, this limit is the described matrix $D_1 A D_2$. If A has support which is not total, this limit cannot be of the form $D_1 A D_2$.

PROOF:

Suppose $B = D_1 A D_2$ and $B' = D_1' A D_2'$ are quaternion doubly stochastic matrix.

$$D_1 = diag\ (x_1, x_2, \ldots\ldots\ldots x_n)$$
$$D_2 = diag\ (y_1, y2, \ldots\ldots\ldots y_n)$$
$$D_1' = diag\ (x_1', x_2', \ldots\ldots x_n')$$
$$D_2' = diag\ (y_1' \ldots\ldots\ldots y_n')$$

$$p_i = x_i' / x_i$$

$$qj = yj' / yj'$$

$$\sum_p x_i a_{ij} y_j = 1 \sum_j x_i a_{ij} y_j = 1$$

$$\sum_{i'} x_i a_{ij} y_j = 1 \sum_j x_i' a_{ij} y_j' = 1$$

$$\sum_{i=1}^{n} \min x_i a_{ij} y_j \leq 1 \leq \sum_{j=1}^{n} \max x_i a_{ij} y_j$$

$$\sum_i^n p_i x_i a_{ij} q_j y_j = 1; \sum_j^n p_i x_i a_{ij} q_j y_j = 1$$

$$Let\ Ej = \{i / a_{ij} > 0\} Fi = \{j / a_{ij} > 0\}$$

$\boldsymbol{m} = \left\{ \boldsymbol{i} / \boldsymbol{p_i} = \min_i \boldsymbol{P_i} = \underline{\boldsymbol{P}} \right\},$

$\boldsymbol{M} = \left\{ \boldsymbol{j} / \boldsymbol{q_j} = \max_j \boldsymbol{q_j} = \overline{\boldsymbol{q}}_1 \right\}$

Assume,

$\boldsymbol{i_o} \in \boldsymbol{m}, \boldsymbol{j_o} \in \boldsymbol{M}$, **Then** $q_o \left(\sum_i P_i x_i a_i j_o y_{j_o} \right) - 1 \leq p_{i_o}^{-1}$

$\boldsymbol{P_{i_o}} \geq \boldsymbol{q_{j_o}}^{-1}, \boldsymbol{q_{j_o}} = \boldsymbol{P_{i_o}}^{-1} = \underline{\boldsymbol{P}^{-1}}$

$\boldsymbol{P_i} = \underline{\boldsymbol{P}}$ ***when*** $\boldsymbol{i} \in \boldsymbol{E_{j_o}}$.

Thus

$\bigcup_{j \in M} Ej \subseteqq M$ **and it follows that A**$\left[\text{m/M}\right]$ **= 0. In the same way**

$\boldsymbol{P_{i_o}} = \boldsymbol{q_{i_o}}^{-1}$ **is possibly only if** $\boldsymbol{q_j} = \overline{\boldsymbol{q}}$ **for all** $\boldsymbol{j} \in \boldsymbol{F_{i_o}}$.

Hence $\boldsymbol{q_j} = \overline{\boldsymbol{q}}$, **when** $\boldsymbol{j} \in \boldsymbol{F_i}$ **and i** $\in$ **m.**

Thus $\bigcup_{i \in m} Fi \subseteqq$ **and it follows that A [m/M] = 0.**

On m x M, Piqj = $\underline{p}\,\overline{q}$ **and it follows that B [m/M] = B' [m/M] is quaternion doubly stochastic. In particular m and M must have the same size.**

If A is fully indecomposable, A(m/M] and A (m/M) thus cannot exist. In such a case A = A(m/M]. Thus $D_1AD_2 = D_1'AD_2'$ **and** D_1 **and** D_2 **are themselves unique upto a scalar multiple.**

If A(m/M] and [m/M) exists, B(m/M) and B'(m/M) exist and are each quaternion doubly stochastic matrices of order less than n. Further more B(m/M) = D_1'' **A(m/M)** D_2'' **and B'(m/M) =** D_1'''**A(m/M)**D_2'''

Where the D's are diagonal matrices with positive main diagonals. The argument may be repeated on these submatrices until $D_1AD_2 = D_1'AD_2'$ **is established.**

Lemma - 1

If A $\in$ $\mathbf{H}^{n \times n}$ **is a row stochastic quaternion matrix and** $\beta_1, \beta_2, \ldots \beta_n$ **are columns of A, then** $\prod_{k=1}^{n} \beta_k \leqq 1$, **with equality only if each** $\beta_k = 1$.

Proof:

Let A have column sums $\beta_1, \ldots\ldots\beta_n$ **of course, each** $\beta_k \geqq 0$ **and** $\sum_{k=1}^{n} \beta_k = n$.

By arithmetic geometric mean inequality $\prod_{k=1}^{n} \beta_k \leqq \left[\left[1/N \right] \sum_{k=1}^{n} \beta_k \right]^n = 1$

with equality occuring only if each

$\beta_k = 1$

$$\min \prod_{k=1}^{n} \beta_k \le \max \left[(1/N) \sum_{k=1}^{n} \beta_k \right]^n = 1$$

Lemma : 2

Let A = (a_{ij}) be an nxn non-negative quaternion matrix with total support and suppose that if 1 $\le$ i, j $\le$ N, $\{x_{i,n}\}$ and $\{y_{j,n}\}$ are positive sequences such that x_i, ny_j, n converges to a positive limit for each i, j such that aij $\neq$ 0 then there exist convergent positive sequences $\{x'_{i,n}\}, \{y'_{j,n}\}$ with positive limits such that $x'_{i,n}\ y'_{j,n} = x_{i,n} y_{j,n}$ for all i,j,n.

PROOF:

A is fully indecomposable.

Let

$E^{(1)} = \{1\}$

$F^{(1)} = \{ j / a_{ij} \succ 0 \}$

$E^{(s)} = \{ i \notin \bigcup_{k=1}^{s-1} \in^{(k)}$ /for some

$j \in F^{(s-1)}$, $a_{ij} \succ 0$ }

$F^{(S)} = \{ j \notin \bigcup_{k=1}^{s-1} F^{(k)}$ / for some

$i \in E^{(s)}$, $a_{ij} \succ 0$ } when s $\succ$ 1.

The sets $E^{(s)}$ and $F^{(s)}$ are void for sufficiently large S.

Define E = U_k $E^{(k)}$ and F = U_k $F^{(k)}$.

Since A has total support, the first row of A contains a nonzero element; Thus $F^{(1)}$ is non-void. Since $F^{(1)} \subseteqq$ F, F is nonvoid. Also since $\{1\} = E^{(1)} \subseteqq$ E is nonvoid.

Suppose E is a proper subset of $\{1, 2, ...n\}$. Pick i $\notin E$, $j \in F$. Then j $\in F^{(s)}$ for some S. Since i $\notin$ E, certainly then it could not be that $a_{ij} \succ 0$ for then i $\in E^{(s+1)} \subseteqq E$, a contraolication. Hence i $\notin E$, $j \in F \Rightarrow a_{ij} = 0$,

ie., A(E/F] = 0

F $\neq$ $\{1, 2, ...n\}, A[E/F) = 0.$

Define an nxn matrix H = (h_{ij}) as follows. If a_{ij} = 0, set h_{ij} = 0, if $a_{ij} \neq 0$ and a_{ij} lies on t positive diagonals in A, set h_{ij} = t/τ where τ is the total number of positive diagonals in A.

Then H is quaternion doubly stochastic and h_{ij} = 0 if and only if a_{ij} = 0. Suppose E contains u elements and F contains V elements.

Since H (E/F] = 0, $\sum_{i \in F} \sum_{j \in F} h_{ij} = v$, since either F = $\{1, ...n\}$ or H[E/F) = 0, $\sum_{i \in E} \sum_{j \in F} h_{ij} = u$, Thus E and F have the same number of elements. But E and F cannot be proper subsets of $\{1,.....n\}$ if A is assumed to be fully indecomposable.

Thus

E = F = $\{1,.......n\}$

Define $x'_{i,n} = x_{i,n} y_{j,n}$ and

$y'_{j,n} = x_{i,n} y_{j,n} = x_{i,n} y_{j,n}$ for all i,j,n.

Then $x'_{i,n}\, y'_{j,n} = x_{i,n} y_{j,n}$ for all i,j,n.Since $x'_{1,n} = 1$ for all n.

Certainly $x'_{i,n} \to 1$. For j $\in F^{(1)}$, $y'_{j,n} = x'_{i,n}\, y'_{j,n} = x_{i,n} y_{j,n}$ has a non-negative limit.

Inductively suppose that is known that $x'_{i,n}$ and $y'_{j,n}$ converge to positive limits when $i \in \bigcup_{k=1}^{s-1} E^{(k)}$ and $j \in \bigcup_{k=1}^{s-1} F^{(k)}$. For it $E^{(s)}$ there is a $j_{s-1} \in F^{(s-1)}$ such that $a_{ij\ s-1} \succ 0$. Thus $x'_{i,n} = x'_{i,n}\, y^{1}_{s-1,n} / y'_{j\ s-1,n} = x_{i,n} y_{j\ s-1,n} / y'_{j\ s-1,n}$ has a positive limit.

Then for $j \in F^{(s)}$ there is $a_{is} \in E^{(s)}$ such that $a_{is} j \succ 0$. When $y'_{j,n} = x'_{is,n}\, y'_{j,n} / x'_{is,n} = x_{is,n} y_{j,n} / x'_{is,n}$ has a positive limit.

This completes the Proof in cas A is fully indecomposable quaternion doubly stochastic matrices.

If A is not fully inclecomposable, then neither is the corresponding quaternion doubly stochastic matrix H. This means that there exist Permutations P and Q such that PHQ=$H_1 \oplus ...... \oplus H_g$ where each H_k is quaternion doubly stochastic and fully indecomposable. Thus also PAQ = A1$\oplus$.........$\oplus A_g$. Where each A_k has total support and is fully indecomposable.

Define an iteration on A as follows

Let $x_{i,0} = 1, y_{j,0} = \left(\sum_{i=1}^{N} aij\right)^{-i}$ and set $x_{i,n+1} = \propto^{-1}_{i,n}\ x_{i,n} y_{j,n+1} = \beta^{-1}{}_{j,n} y_{j,n}$.

$\propto_{i,n} = \sum_{j=1}^{n} x_{i,n}\ a_{ij} y_{j,n},\ \beta_{j,n} = \sum_{i='}^{n} \propto_{i,n}{}^{-1}\ x_{i,n} \propto_{ij}\ y_{j,n},$

$i = 1,.....n,\ j = 1,.......n,\ n = 0,1,......$

Note that $(x_{i,n} a_{ij}\ y_{j,n})$ is column stochastic and $(x_{i,n+1}\ a_{ij}\ y_{j,n})$ is row stochastic.

$$y_{j,n} = \left(\sum_{i=1}^{n} x_{i,n}\ a_{ij}\right)^{-1} \leqq x_{io}{}^{-1} a_{io,n}{}^{-1} j \leqq x_{io,n}{}^{-1}\ a^{-1}.$$

Where $i_0$is such that $a_{ij} \succ 0$ and a is the minimal positive a_{ij}. Thus
$x_i,_n\, y_j, n \leqq a^{-1}\ if\ a_{ij} \succ 0.$

Let A have a positive diagonal corresponding to a permulatation σ, and Set

$$S_n = \prod_{i=1}^{n} x_{i,n} y_{\sigma(i)}, n \ and$$

$$S_n^{\ 1} = \prod_{i=1}^{n} x_{i,n+1}\ y_{\sigma(i),n}$$

$$S_n \leqq s_n^{\ 1} \leqq s_n + 1 \leqq a^{-n}.$$

Thus $S_n \to L$ and $S_n\,' \to L.$
$where\ o \prec L \leqq a^{-n}.$

$\prod_{j=1}^{n} \beta_j,_n = {}^{s_n'}\!/_{s_n+1} \to 1.$ This is impossible unless each $\beta_j,_n \to 1.$ Since $\prod_{k=1}^{n} \beta_k$ has a unique maximal value of 1. Only when $\beta_1 = ...\beta_n = 1$. Similarly each $\propto_{i,n} \to 1.$ A_n is the n^{th} matrix in the iteration and that $A_{nk} \to B$ and $A_{mk} \to C$. Observe that for any given pair i, j $\beta_{ij} \neq 0 \Leftrightarrow c_{ij} \neq 0$. For any permutations σ,

$\prod_{i=1}^{n} b_i, \sigma_{(i)} = \prod_{i=1}^{n} c_i, \sigma_{ci} = L \prod_{i=1}^{n} a_i, \sigma_{(i)}.$ Thus certainly $b_{ij} \neq 0 \Rightarrow c_{ij} \neq 0$ for suppose $b_{io} j_o \neq 0$ then $b_{io} j_o$ leis on a positive diagonal.

The corresponding diagonal in c would have a positive product. Thus $c_{i0}j_0 \neq 0$ in the same way $c_{ij} \neq 0 \Rightarrow b_{ij} \neq 0$. If in addition A has total support then $a_{ij} \neq 0 \Leftrightarrow b_{ij} \neq 0 \Leftrightarrow c_{ij} \neq 0$.

By construction there exist matrices D^*_1, k = diag $\left(X_{1,k},.........X_{n,k}\right)$ *and*

$D^*_{2,k}$ = diag $\left(Z_{1,k},.........Z_{n,k}\right)$

$A_{mk} = D^*_{1,k}\ A_{mk}\ D^*_{2,k}$. For $b_{ij} \succ 0$,

$X_{i,k}\ Z_{j,k} \rightarrow c_{ij}b_{ij}^{-1}$. By lemma 2, there exist a positive sequences $\{X'_{i,k}\}$ and $\{Z'_{j,k}\}$ coverging to a positive limits such that $X'_{i,k}\ Z'_{j,k} = X_{i,k}\ Z_{j,k}$ for all i, j, k

$D_1^* = \lim_{k\to\infty} diag\ (X_{1,k}'.....X_{n,k})$

$D_2^* = \lim_{k\to\infty} diag\ (Z_{1,k}',.....Z_{n,k})$

then C = D_1*B D_2*. By the uniquences part of the theorem B = C. It follows that the iteration converges.

Suppose A has total support. Let $D_{1,n} = diag\ (X_{1,n},.....X_{n,n})$ and

$D_{2,n} = diag\ (y_{1,n},.....y_{n,n})$. Then B = $\lim_{n\to\infty} diag\ (D_{1,n}\ A_{2,n})$ exists and

$b_{ij} \neq 0 \Leftrightarrow a_{ij} \neq 0$. when $a_{ij} \succ 0$, $x_{i,n}y_{i,n} \rightarrow b_{ij}a_{ij}^{-1}$. By lemma 2 there are convergent positive sequences $\{x'_{i,n}\}\{y'_{i,n}\}$ with positive limits such that $x_{i,n}'\ y_{j,n}' = x_{i,n}\ y_{j,n}$ for all i, j, n.

Let $D_1 = \lim_{n\to\infty} diag\ (x'_{1,n},.....x_{n,n})$

$D_2 = \lim_{n\to\infty} diag\ (y'_{1,n},.....y_{n,n})$

Then B = $D_1 A\ D_2$

Finally we observe that if A has support which is not total, then by Birkhoff's theorem there is a non-zero in the iteration. In fact every non-zero element in A which is not on a positive diagonal must do so. If the limit matrix could be put in the form $D_1 A\ D_2$ then some term $x_i a_{ij} y_j = 0$ where $a_{ij} \succ 0$. But then either $x_i = 0$ or $y_j = 0$. The former leads to a row of zeros and the latter to a column of zeros in $D_1 A\ D_2$. In either case $D_1 A\ D_2$. In either case $D_1 A\ D_2$ could not be quaternion doubly stochastic matrix.

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