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On Commutativity of Non-Associative Primitive Rings with $(\mathbf{xy})^2 - \mathbf{y}(\mathbf{x}^2\mathbf{y}) \in \mathbf{Z(R)}$

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***Abstract**: In this section we have proved that an associative semi prime ring in which $(xy)^2 - yx^2y$ is central, is commutative. In this section, we prove a similar result for non-associative primitive rings. **Ram Awatar** [6] generalized **Gupta's** [5] result and proved that if R is an associative semi prime ring in which $xy^2x - yx^2y$ is central, then R is commutative. In this section we show that if R is an alternative prime ring in which$(xy^2)x - (yx^2)y$is central, then R is commutative.*

***Key Words**: Primitive Ring, Center, Commutativity, Alternative Prime Ring.*

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I. Introduction

The study of associative and non- associative rings has evoked great interest and assumed importance. The results on associative and non- associative rings in which one does assume some identities in the center have been scattered throughout the literature.

Many sufficient conditions are well known under which a given ring becomes commutative. Notable among them are some given by **Jacobson, Kaplansky** and Herstein. Many Mathematicians of recent years studied commutativity of certain rings with keen interest. Among these mathematicians **Herstein, Bell, Johnsen, Outcalt, Yaqub, Quadri and Abu-khuzam** are the ones whose contributions to this field are outstanding.

II. Preliminaries

Non–Associative Ring:
If R is an abelian group with respect to addition and with respect to multiplication R is distributive over addition on the left as well as on the right.
For every elements x, y, z of $\mathrm{R}(\mathrm{x}+\mathrm{y})\mathrm{z} = \mathrm{xz} + \mathrm{yz}, \mathrm{z}(\mathrm{x}+\mathrm{y}) = \mathrm{zx} + \mathrm{zy}$
Alternative rings, Lie rings and Torsion rings are best examples of these non–associative rings.

Alternative Ring For every x and y in R is$(\mathrm{xx})\mathrm{y} = \mathrm{x}(\mathrm{xy}), \mathrm{y}(\mathrm{xx}) = (\mathrm{yx})\mathrm{x}$ then R is said to be alternative ring on the left as well as left.

Commutator:
For every x, y in a ring R satisfying $[\mathrm{x},\mathrm{y}] = \mathrm{xy} - \mathrm{yx}$ then [x,y] is called a commutator

Commutative Ring:
For every x, y in a ring R if $\mathrm{xy} = \mathrm{yx}$ then R is called a commutative ring.
Non–commutative ring is split from the commutative ring, i.e., R is not commutative with respect to multiplication. i.e., we cannot take $xy = yx$ for every x, y in R as an axiom.

Prime Ring:
A ring R is called a prime ring if whenever A and B are ideals of R such that AB = 0 then either A = 0 or B = 0.

Primitive Ring:
A ring R is defined as primitive in case it possesses a regular maximal right ideal, which contains no two–sided ideal of the ring other than the zero ideal.
Torsion–Free ring:
If R is m-torsion free ring, then mx=0 implies x=0 for positive integer m and x is in R.

Center:

In a ring R, the center denoted by Z(R) is the set of all elements $x \in R$ such that xy=yx for all $x \in R$. It is important to note that this definition does not depend on the associative of multiplication and in fact, we shall have occasion to deal with derivation of non-associative algebras.

III. Main Results

Theorem 1 : Let R be a non-associative primitive ring with unity satisfying$(xy^2) - y(x^2y) \in Z(R)$ for all x, y in R Then R is a commutative.

Proof : By hypothesis $(xy)^2 - y(x^2y) \in Z(R)$ …1.1

for all x, y in R

Replacing x by x+1 in 1.1, we get $((x+1)^2y^2 - y((x+1)^2y) \in Z(R)$.

i.e.,$(xy+y)^2 - y(x^2y + 2xy + y) \in Z(R)$

Using 1.1, we obtain $(xy)y - y(xy) \in Z(R)$. ...1.2

Now replacing y by y+1 in 1.2,and using 1.2 we get $xy - yx \in Z(R)$.

If R is a primitive ring then R has a maximal right ideal which contains no non- zero ideal of R. Consequently, we obtain (xy– yx)R = 0. This further yields $xy - yx = 0$ due to primitivity of R.

Hence R is commutative. □

Theorem 2: Let R be an alternative prime ring$(xy^2)x - (yx^2)y \in Z(R)$ for all x, y in R .Then R is commutative.

Proof: First we shall prove that Z(R) $\neq$ (0).

Let us suppose that Z(R)=(0).

Hence by hypothesis,$(xy^2)x = (yx^2)y$, for all x, y in R. ...2.1

Replacing y by y + y2 in 2.1,

we obtain$(x(y^2 + y^4 + 2y^3))x = (yx^2 + y^2x^2)(y + y^2)$

i.e. ,$(xy^2)x + (xy^4)x + 2(xy^3)x = (yx^2)y + (yx^2)y^2 + (y^2x^2)y^2$

i.e.,$2(xy^3)x = (y^2x^2)y + (yx^2)y^2$ ...2.2

Since$(y^2x^2)y = \big(y(yx^2)\big)y = y((yx^2)y = y\big((xy^2)x\big) = \big((yx)y^2\big)x = (yx)(y^2x)$

and$(yx^2)y^2 = \big((yx)x\big)y^2 = (yx)(xy^2)$.

Hence 2.2 reduced to, $2(xy^3)x= (yx)(y^2x+xy^2)$ ...2.3

If R is not 2-torsion free, 2.3become $(yx)(y^2x + xy^2) = 0$.

With x=x+y, this gives$(yx + y^2)(y^2x + y^3 + xy^2 + y^3) = 0$..

i.e. ,$y^2(y^2x + y^2) = 0$. ...2.4

Put x=rx in 2.4, then we get$y^2(y^2(rx) + (rx)y^2) = 0$.. ...2.5

Since $y^2(y^2r) = y^2(ry^2)$,

From 2.4 and 2.5, we have $y^2\big(r(y^2x + xy^2)\big) = 0$.we write this as $y^2R(y^2x + xy^2) = 0$

Since R is prime, either $y^2 = 0$ or$y^2x + xy^2 = 0$.

i.e., $y^2R \in Z(R) = 0$ $y^2 \in Z(R) = 0$.

Thus in either case $y^2 = 0$ for every y in R .

If R is 2- torsion free, we replace y by y + y^3 in 2.1 and get $2(xy^4)x = (y^3x^2)y + (xy^2)y^3$

$$2(y^2x^2)y^2 = y^2\big((yx^2)y\big) + \big((yx^2)y\big)y^2 = y^2\big((xy^2)x\big) + \big((xy^2)x\big)y^2.$$

We write this as$(y^2x^2)y^2 - y^2\big((xy^2)x\big) = \big((xy^2)x\big)y^2 - (y^2x^2)y^2$ or $(y^2x)(xy^2 - y^2x) = (xy^2 - y^2x)y^2$

We replacing x by x+y: Then we get$y^3(xy^2 - y^2x) = (xy^2 - y^2x)y^3$ ...2.6

for all x, y in R.

Let I_y^2 be the inner derivation by y^2

i.e.$x - -\geq xy^2 - y^2x$, and I_y^3 be the inner derivation by y^3.

Then 2.6becomes $I_y^3 I_y^2(x) = 0$.

Thus the product of these derivation is again a derivation. Then by the lemma we can conclude that either y^2 *or* y^3 in Z (R), i.e., y^2 *or* y^3 is zero.

If $y^3 = 0$, $y^3 = 0$, then 2.2, becomes $(y^2x^2)y + (yx^2)y^2 = 0$..

Substituting x+y for x, we get $(y^2x^2 + y^3 + 2y^2(xy))y + (yx^2 + y^3 + 2y(xy))y^2 = 0$ *ie* $2(y^2x)y^2 + 2(y(xy^3)y^2) = 0$.

Then we get $2(y^2x)y^2 = 0$ *or* $(y^2x)y^2 = 0$ *or* $(y^2R)y^2 = 0$ *then* $y^2 = 0$

Thus if Z (R) = (0), then $y^2 = 0$ for every y in R.

Then 0 = $(x+y^2)x$ = (x y)x or xRx = 0

Then x=0 or R=0, a contradiction. Therefore Z(R) $\neq$ (0).

Taking $\lambda \neq 0$ in Z(R) and let $x = x + \lambda$ in $(xy^2)x - (yx^2)x - (yx^2)y$ in Z(R), we get

$\lambda(xy^2 - 2(yx)y + y^2(x))$ in Z(R).

Since R is prime, we must have $xy^2 - 2(yx)y + y^2x$ in Z(R). ...2.7

If λ a is in Z(R), then $\lambda ab - b\lambda a = 0 = \lambda(ab - ba)$.

Then, R λ (ab-ba)=0= λ R(ab-ba) and since $\lambda \neq 0$, we have

ab - ba=0, i.e., is in Z(R).

In 2.7, we let x=xy and get

, $xy^2 - 2(yx)y + (y^2x)y$ in Z(R), then y is in Z(R),

Unless, $xy^2 - 2(yx)y + y^2x = 0$.for every x in R, and

y is in Z(R), then , $xy^2 - 2(yx)y + y^2x$ is still zero.

Therefore ,$xy^2 + y^2x = 2(yx)y$, for every x ,y in R. ...2.8

If R is 2- torsion free, then R is commutative by lemma [38, P.5].

If R is not 2-torsion free, then 2.8. become $xy^2 + y^2x = 0$ or y^2 is in Z(R) for every y in R Then $(x + y)^2 = x^2 + y^2 + xy + yx$ is in Z(R)

i.e., x y+ y x is in Z(R).

Let x=x y and get (x y+ y x)y is in Z(R).

Then y is in Z(R), unless xy + yx= 0, which also means y is in Z(R)

Thus Z(R)=R and R is commutative. □

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