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On Heron Triangles

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Abstract: *Different set of formulas for integer heron triangles are obtained.*
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I. INTRODUCTION

The numbers that can be represented by a regular geometric arrangement of equally spaced points are called the polygonal numbers or Figurate numbers. Mathematicians from the days of ancient Greeks have always been interested in the properties of numbers that can be arranged as a triangle, which is a three-sided polygon. There are many different kinds of triangles of which heron triangle is one. A heron triangle is a triangle having rational side lengths and rational area [1]. One may refer [2, 3] for integer heron triangles. If a, b, c are the sides of the heron triangle then the triple (a, b, c) is known as Heron triple. The Indian mathematician Brahmagupta derived the parametric version of integer heron triangles [4-6]. In [7], Charles Fleenor illustrates the existence of Heron triangles having sides whose lengths are consecutive integers. In [8], the general problem of Heron triangles with sides in any arithmetical progression is discussed. The above results motivated us to search for different set of formulas for integer heron triangles which is the main thrust of this paper.

This paper consists of three sections 1, 2 and 3. In section 1, we illustrate the process of obtaining different set of formulas for integer heron triangles. In section 2, we present heron triangles with sides in Arithmetic progression and it seems that they are not presented earlier. Section 3 deals with the different sets of isosceles heron triangles.

II. METHOD OF ANALYSIS

2.1. Section: 1 Formulas for integer heron triangles

Let the three positive integers a, b, c be the lengths of the sides BC, CA, AB respectively of the heron triangle ABC. Consider the cosine formula given by

$$a^2 = b^2 + c^2 - 2bc\cos A \qquad (1.1)$$

Let $\cos A = \dfrac{\alpha}{\beta}, \quad \beta > \alpha > 0$ (1.2)

where $\beta^2 - \alpha^2 = D^2 \quad (D > 0)$ (1.3)

Substitution of (1.2) in (1.1) gives

$$2bc\alpha = \beta\left(b^2 + c^2 - a^2\right) \qquad (1.4)$$

Introducing the linear transformations

$$b = 2X + 2\alpha T, \quad c = 2\beta T, \quad a = 2A \qquad (1.5)$$

in (1.4), it is written as

$$A^2 = X^2 + D^2T^2 \qquad (1.6)$$

which is in the form of well-known Pythagorean equation satisfied by

$$X = 2mn, \quad DT = m^2 - n^2, \; A = m^2 + n^2, \; m > n > 0 \qquad (1.7)$$

Choosing $m = DM$ and $n = DN$ in (1.7), we have

$$\left.\begin{aligned} X &= 2D^2MN \\ T &= D\left(M^2 - N^2\right) \\ A &= D^2\left(M^2 + N^2\right), \quad M > N > 0 \end{aligned}\right\} \qquad (1.8)$$

Substituting (1.8) in (1.5), the values of a, b, c are given by

$$a = 2D^2(M^2 + N^2)$$
$$b = 4D^2MN + 2\alpha D(M^2 - N^2)$$
$$c = 2\beta D(M^2 - N^2)$$

The area of the triangle ABC is given by

$$H = 2D^3(M^2 - N^2)(\alpha(M^2 - N^2) + 2DMN) \quad (1.9)$$

A few numerical examples are given in Table: 1.1 below:

Table: 1.1 Numerical examples

α	β	D	M	N	Heron Triple	Area
4	5	3	2	1	(90, 144, 90)	3888
12	13	5	3	2	(650, 1200, 650)	150000
24	25	7	4	2	(1960, 5600, 4200)	3292800
15	17	8	3	1	(1280, 2688, 2176)	1376256
8	10	6	2	1	(360, 576, 360)	62208
24	26	10	3	2	(2600, 4800, 2600)	2400000
45	51	24	3	1	(11520, 24192, 19584)	111476736

When $\frac{\beta + D}{\beta - D} = \left(\frac{M}{N}\right)^2$ then the integer heron triangle is isosceles and its perimeter is a perfect square.

It may be noted that, (1.6) is also satisfied by

$$X = m^2 - n^2,\ DT = 2mn,\ A = m^2 + n^2,\ m > n > 0 \quad (1.10)$$

Case: (i)

The choice $m = DM$ in (1.10) gives

$$X = D^2M^2 - n^2$$
$$T = 2Mn$$
$$A = D^2M^2 + n^2$$

The corresponding values of the sides and area H of the triangle ABC are given by

$$a = 2(D^2M^2 + n^2)$$
$$b = 4\alpha Mn + 2(D^2M^2 - n^2)$$
$$c = 4\beta Mn$$
$$H = 4DMn(D^2M^2 + 2\alpha Mn - n^2)$$

A few numerical examples are given in Table: 1.2 below:

Table: 1.2 Numerical examples

α	β	D	M	n	Heron Triple	Area
4	5	3	2	1	(74, 102, 40)	1224
12	13	5	3	2	(458, 730, 312)	43800
24	25	7	4	2	(1576, 2328, 800)	260736
15	17	8	3	1	(1154, 1330, 204)	63840
8	10	6	2	1	(290, 350, 80)	8400
24	26	10	3	2	(1808, 2368, 624)	284160
45	51	24	3	1	(10370, 10906, 612)	1570464

Case: (ii)

The choice $n = DN$ in (1.10) gives

$$X = m^2 - D^2N^2$$
$$T = 2mN$$
$$A = m^2 + D^2N^2$$

The corresponding values of the sides and area H of the triangle ABC are given by

$$a = 2(m^2 + D^2N^2)$$
$$b = 4\alpha mN + 2(m^2 - D^2N^2)$$
$$c = 4\beta mN$$
$$H = 4DmN(m^2 + 2\alpha mN - D^2N^2)$$

A few numerical examples are given in Table: 1.3 below:

Table: 1.3 Numerical examples

α	β	D	m	N	Heron Triple	Area
4	5	3	4	1	(50, 78, 80)	1872
12	13	5	11	2	(442, 1098, 1144)	241560
24	25	7	8	1	(226, 798, 800)	89376
15	17	8	17	2	(1090, 2106, 2312)	1145664
8	10	6	7	1	(170, 250, 280)	21000
24	26	10	11	1	(442, 1098, 1144)	241560
45	51	24	50	2	(9608, 18392, 20400)	88281600

Case: (iii)

The choice $m = DM$ and $n = DN$ in (1.10) gives

$$X = D^2\left(M^2 - N^2\right)$$
$$T = 2DMN$$
$$A = D^2\left(M^2 + N^2\right)$$

The corresponding values of the sides and area H of the triangle ABC are given by

$$a = 2D^2\left(M^2 + N^2\right)$$
$$b = 4\alpha DMN + 2D^2\left(M^2 - N^2\right)$$
$$c = 4\beta DMN$$
$$H = 4D^3 MN\left(DM^2 + 2\alpha MN - DN^2\right)$$

A few numerical examples are given in Table: 1.4 below:

Table: 1.4 Numerical examples

α	β	D	M	N	Heron Triple	Area
4	5	3	2	1	(90, 150, 120)	5400
12	13	5	3	2	(650, 1690, 1560)	507000
24	25	7	4	2	(1960, 6552, 5600)	5136768
15	17	8	3	1	(1280, 2464, 1632)	946176
8	10	6	2	1	(360, 600, 480)	86400
24	26	10	3	2	(2600, 6760, 6240)	8112000
45	51	24	3	1	(11520, 22176, 14688)	76640256

Also, equation (1.6) is represented as the system of double equations as shown below:

System: 1 $A + X = D^2 T$, $A - X = T$

System: 2 $A + X = D^2$, $A - X = T^2$

System: 3 $A + X = DT^2$, $A - X = D$

Consider System: 1. The corresponding values of the sides and area H of the triangle ABC are given by

$$a = 2\left(D^2 + 1\right)k$$
$$b = 4\alpha k + 2\left(D^2 - 1\right)k$$
$$c = 4\beta k$$
$$H = 4k^2 D\left(D^2 + 2\alpha - 1\right)$$

A few numerical examples are given in Table: 1.5 below:

Table: 1.5 Numerical examples

α	β	D	k	Heron Triple	Area
4	5	3	2	(40, 64, 40)	768
12	13	5	3	(156, 288, 156)	8640
24	25	7	2	(200, 384, 200)	10752
15	17	8	3	(390, 558, 204)	26784
8	10	6	1	(74, 102, 40)	1224
24	26	10	2	(404, 588, 208)	23520
45	51	24	1	(1154, 1330, 204)	63840

Now consider System: 2. In this case, there are two sets of values for the sides and area H of the triangle ABC given by

Set: 1

$$a = 4\left(d^2 + t^2\right)$$

$$b = 4\alpha t + 4\left(d^2 - t^2\right)$$

$$c = 4\beta t$$

$$H = 16td\left(d^2 + \alpha t - t^2\right)$$

A few numerical examples are given in Table: 1.6 below:

Table: 1.6 Numerical examples

α	β	D	d	t	Heron Triple	Area
8	10	6	3	2	(52, 84, 80)	2016
24	26	10	5	3	(136, 352, 312)	21120
45	51	24	12	2	(592, 920, 408)	88320
15	17	8	4	3	(100, 208, 204)	9984

Set: 2

$$a = 4d^2 + 4d + 4t^2 + 4t + 2$$

$$b = 4d^2 + 4d - 4t^2 - 4t + 4\alpha t + 2\alpha$$

$$c = 4\beta t + 2\beta$$

$$H = (2d+1)(2t+1)\left((2d+1)^2 + 2\alpha(2t+1) - (2t+1)^2\right)$$

A few numerical examples are given in Table: 1.7 below:

Table: 1.7 Numerical examples

α	β	D	d	t	Heron Triple	Area
4	5	3	1	2	(34, 24, 50)	360
12	13	5	2	3	(74, 144, 182)	5040
24	25	7	3	2	(74, 264, 250)	9240

Consider System: 3. In this case, there are two sets of values for the sides and area H of the triangle ABC given by

Set: 3

$$a = 2\left(T^2 + 1\right)k$$

$$b = 2\alpha T + 2k\left(T^2 - 1\right)$$

$$c = 2\beta T$$

$$H = 4kT\left(kT^2 + \alpha T - k\right)$$

A few numerical examples are given in Table: 1.8 below:

Table: 1.8 Numerical examples

α	β	D	k	T	Heron Triple	Area
8	10	6	3	2	(30, 50, 40)	600
24	26	10	5	3	(100, 224, 156)	6720
45	51	24	12	2	(120, 252, 204)	12096
15	17	8	4	3	(80, 154, 102)	3696

Set: 4

$$a = 2D\left(2k^2 + 2k + 1\right)$$

$$b = 4Dk(k+1) + 2\alpha(2k+1)$$

$$c = 2\beta(2k+1)$$

$$H = 2D(2k+1)\left(2Dk(k+1) + \alpha(2k+1)\right)$$

A few numerical examples are given in Table: 1.9 below:

Table: 1.9 Numerical examples

α	β	D	k	Heron Triple	Area
4	5	3	2	(78, 112, 50)	1680
12	13	5	2	(130, 240, 130)	6000
15	17	8	3	(400, 594, 238)	33264
8	10	6	4	(492, 624, 180)	33696

2.2. Section: 2 Formulas for integer heron triangles with sides in Arithmetic Progression

Let a, d be two non-zero distinct positive integers such that $a > d > 0$. Then the triple $(a-d, a, a+d)$ is in Arithmetic progression. Consider the above triple to be an integer Heron triangle with area H. Employing the Heron's formula for area, we have

$$H = \frac{a}{4}\sqrt{3\left(a^2 - 4d^2\right)} \tag{2.1}$$

The choice

$$a = 2A \tag{2.2}$$

in (2.1) leads to

$$H = A\sqrt{3\left(A^2 - d^2\right)} \tag{2.3}$$

To eliminate the square-root on the R.H.S of (2.3), assume

$$A^2 - d^2 = 3g^2 \tag{2.4}$$

Equation (2.4) is represented as the system of double equations as shown below:

System: 1 $A + d = g^2, \quad A - d = 3$

System: 2 $A + d = 3g^2, \quad A - d = 1$

Consider System: 1.The corresponding sides and area H of the integer heron triangle are given by

Sides: $2k^2 + 2k + 5, \quad 4k^2 + 4k + 4, \quad 6k^2 + 6k + 3$

$H = (6k+3)\left(2k^2 + 2k + 2\right)$

A few numerical examples are given in the Table 2.1 below:

Table: 2.1 Numerical examples

k	Heron Triple	Area	Perimeter
2	(17, 28, 39)	210	84
3	(29, 52, 75)	546	156
5	(65, 124, 183)	2046	372
4	(45, 84, 123)	1134	252

Now consider System: 2. In this case, the corresponding sides in A.P of the integer heron triangle are

$6k^2 + 6k + 3, \quad 12k^2 + 12k + 4, \quad 18k^2 + 18k + 5$

with area $H = (6k+3)\left(6k^2 + 6k + 2\right)$

A few numerical examples are given in the Table: 2.2 below:

Table: 2.2 Numerical examples

k	Heron Triple	Area	Perimeter
2	(39, 76, 113)	570	228
3	(75,148, 221)	1554	444
5	(183, 364, 545)	6006	1092
4	(123, 244,365)	3294	732

One may write (2.4) as

$$d^2 + 3g^2 = A^2 * 1 \tag{2.5}$$

Assume

$$A = \alpha^2 + 3\beta^2, \quad \alpha, \beta > 0 \tag{2.6}$$

Write 1 as

$$1 = \frac{\left(1 + i\sqrt{3}\right)\left(1 - i\sqrt{3}\right)}{4} \tag{2.7}$$

Substituting (2.6), (2.7) in (2.5) and employing the method of factorization, define

$$d + i\sqrt{3}g = \frac{1}{2}\left(1 + i\sqrt{3}\right)\left(\alpha + i\sqrt{3}\beta\right)^2 \quad (2.8)$$

Equating real and imaginary parts of (2.8), one obtains

$$\left.\begin{aligned} d &= \frac{1}{2}\left(\alpha^2 - 3\beta^2 - 6\alpha\beta\right) \\ g &= \frac{1}{2}\left(\alpha^2 - 3\beta^2 + 2\alpha\beta\right) \end{aligned}\right\} \quad (2.9)$$

Case: (i)

The choices $\alpha = 2R$, $\beta = 2S$ in (2.9), (2.6) lead to

$$d = 2R^2 - 6S^2 - 12RS$$
$$g = 2R^2 - 6S^2 + 4RS$$
$$A = 4R^2 + 12S^2$$

In view of (2.2), we have

$$a = 8R^2 + 24S^2$$

Hence, the corresponding sides and area H of the integer heron triangle are given by

Sides: $6R^2 + 30S^2 + 12RS$, $8R^2 + 24S^2$, $10R^2 + 18S^2 - 12RS$

$$H = \left(12R^2 + 36S^2\right)\left(2R^2 - 6S^2 + 4RS\right)$$

A few numerical examples are given in the Table 2.3 below:

Table: 2.3 Numerical examples

R	S	Heron Triple	Area	Perimeter
2	1	(78, 56, 34)	840	168
3	2	(246, 168, 90)	4536	504
5	3	(600, 416, 232)	34944	1248
4	2	(312, 224, 136)	13440	672

Case: (ii)

The choices $\alpha = 2R + 1$, $\beta = 2S + 1$ in (2.9), (2.6) lead to

$$d = 2R^2 - 6S^2 - 4R - 12S - 12RS - 4$$
$$g = 2R^2 - 6S^2 + 4R - 4S + 4RS$$
$$A = 4R^2 + 12S^2 + 4R + 12S + 4$$

In view of (2.2), we have

$$a = 8R^2 + 24S^2 + 8R + 24S + 8$$

Hence, the corresponding Heron triple and area H of the integer heron triangle are given by

$(6R^2 + 30S^2 + 12R + 36S + 12RS + 12,\quad 8R^2 + 24S^2 + 8R + 24S + 8,\quad 10R^2 + 18S^2 + 4R + 12S - 12RS + 4)$

$$H = \left(4R^2 + 12S^2 + 4R + 12S + 4\right) * \left(6R^2 - 18S^2 + 12R - 12S + 12RS\right)$$

A few numerical examples are given in the Table 2.4 below:

Table: 2.4 Numerical examples

R	S	Heron Triple	Area	Perimeter
2	1	(150,104,58)	2184	312
3	2	(366, 248, 130)	8184	744
5	3	(780, 536, 292)	51456	1608
4	2	(444, 312, 180)	22464	936

It is worth to note that (2.4) may be written as

$$\frac{A + 2g}{d + g} = \frac{d - g}{A - 2g} = \frac{m}{n}, \quad n \neq 0 \quad (2.10)$$

which is equivalent to the system of double equations

$$nA + (2n - m)g - md = 0$$
$$-mA + (2m - n)g + nd = 0$$

Applying the method of cross multiplication, we have

$$A = 2m^2 + 2n^2 - 2mn$$

$$g = m^2 - n^2$$
$$d = -m^2 - n^2 + 4mn$$

In view of (2.2), we have

$$a = 4m^2 + 4n^2 - 4mn$$

Hence, the corresponding sides in A.P of the integer heron triangle are given by

$5m^2 + 5n^2 - 8mn, \quad 4m^2 + 4n^2 - 4mn, \quad 3m^2 + 3n^2$ with area

$$H = (m^2 - n^2)(6m^2 + 6n^2 - 6mn)$$

A few numerical examples are given in the Table 2.5 below:

Table: 2.5 Numerical examples

m	n	Heron Triple	Area	Perimeter
2	1	(9, 12, 15)	54	36
3	2	(17, 28, 39)	210	84
5	3	(50, 76, 102)	1824	228
4	2	(36, 48, 60)	864	144

2.3. Section: 3 Formulas for integer isosceles heron triangles

2.3.1. Pattern: 1

Let the sides of integer isosceles heron triangle be a, a, b. Employing the Heron's formula for area H, we have

$$H = \frac{b}{4}\sqrt{4a^2 - b^2} \tag{3.1}$$

Case: 1

The choice

$$b = 2B \tag{3.2}$$

in (3.1) leads to

$$H = B\sqrt{a^2 - B^2} \tag{3.3}$$

To eliminate the square-root on the R.H.S of (3.3), assume

$$a^2 - B^2 = \alpha^2 \tag{3.4}$$

which is in the form of well-known Pythagorean equation and is satisfied by

$$a = r^2 + s^2, \quad \alpha = 2rs, \quad B = r^2 - s^2, \quad r > s > 0 \tag{3.5}$$

In view of (3.2), we have

$$b = 2(r^2 - s^2)$$

Hence, the corresponding sides of the integer isosceles heron triangle are given by

$$r^2 + s^2, \quad r^2 + s^2, \quad 2(r^2 - s^2)$$

with area $H = 2rs\,(r^2 - s^2)$

A few numerical examples are given in the Table 3.1 below:

Table: 3.1 Numerical examples

r	s	Heron Triple	Area	Perimeter
2	1	(5, 5, 6)	12	16
3	2	(13, 13, 10)	60	36
5	1	(26, 26, 48)	240	100
4	2	(20, 20, 24)	192	64

It is observed that 6* perimeter is a Nasty number. [9]

Instead of (3.5), equation (3.4) is also satisfied by

$$a = r^2 + s^2, \quad \alpha = r^2 - s^2, \quad B = 2rs, \quad r > s > 0 \tag{3.6}$$

Then the corresponding sides and area of the integer isosceles heron triangle are given by

Sides: $r^2 + s^2, \quad r^2 + s^2, \quad 4rs$

$$H = 2rs\,(r^2 - s^2)$$

A few numerical examples are given in the Table 3.2 below:

Table: 3.2 Numerical examples

r	s	Heron Triple	Area	Perimeter
2	1	(5, 5, 8)	12	18
3	2	(13, 13, 24)	60	50
5	1	(26, 26, 20)	240	72
4	2	(20, 20, 32)	192	72

Note that 2*Perimeter is a perfect square

Case: (ii)

Let $4a^2 - b^2 = z^2$ (3.7)

which is in the form of well-known Pythagorean equation and is satisfied by

$b = 2rs$, $z = r^2 - s^2$, $2a = r^2 + s^2$, $r > s > 0$ (3.8)

The choices

$r = 2R$, $s = 2S$ (3.9)

in (3.8) lead to

$$a = 2\left(R^2 + S^2\right)$$
$$z = 4\left(R^2 - S^2\right)$$
$$b = 8RS$$

Hence, the corresponding sides of the integer isosceles heron triangle are given by

$2\left(R^2 + S^2\right),\quad 2\left(R^2 + S^2\right),\quad 8RS$

with area $H = 8RS\left(R^2 - S^2\right)$

A few numerical examples are given in the Table 3.3 below:

Table: 3.3 Numerical examples

R	S	Heron Triple	Area	Perimeter
2	1	(10, 10, 16)	48	36
3	2	(26, 26, 48)	240	100
5	3	(68, 68, 120)	1920	256
4	2	(40, 40, 64)	768	144

It is observed that 6* perimeter is a Nasty number.

Instead of (3.9), if we choose the choices

$r = 2R + 1$, $s = 2S + 1$ (3.10)

in (3.8) lead to

$$a = 2R^2 + 2S^2 + 2R + 2S + 1$$
$$z = 4R^2 - 4S^2 + 4R - 4S$$
$$b = 2\left(2R + 1\right)\left(2S + 1\right)$$

Then, the corresponding sides and area H of the integer isosceles heron triangle are given by

Sides: $2R^2 + 2S^2 + 2R + 2S + 1,\quad 2R^2 + 2S^2 + 2R + 2S + 1,\quad 2\left(2R + 1\right)\left(2S + 1\right)$

$H = 2\left(2R + 1\right)\left(2S + 1\right)\left(R^2 - S^2 + R - S\right)$

A few numerical examples are given in the Table 3.4 below:

Table: 3.4 Numerical examples

R	S	Heron Triple	Area	Perimeter
2	1	(17, 17, 30)	120	64
3	2	(37, 37, 70)	420	144
5	3	(85, 85, 154)	2772	324
4	2	(53, 53, 90)	756	196

Note that 6*Perimeter is a Nasty number.

Instead of (3.8), equation (3.7) is also satisfied by

$b = r^2 - s^2$, $z = 2rs$, $2a = r^2 + s^2$, $r > s > 0$ (3.11)

In this case, there are two sets of values for the sides and the area H of the integer isosceles heron triangle given by

Set: 1

Sides: $2\left(R^2+S^2\right),\quad 2\left(R^2+S^2\right),\quad 4\left(R^2-S^2\right)$

$H=8RS\left(R^2-S^2\right)$

A few numerical examples are given in the Table 3.5 below:

Table: 3.5 Numerical examples

R	S	Heron Triple	Area	Perimeter
2	1	(10, 10, 12)	48	32
3	2	(26, 26, 20)	240	72
5	3	(68, 68, 64)	1920	200
4	2	(40, 40, 48)	768	128

It is to be noted that 2*Perimeter is a perfect square

Set: 2

Sides: $2R^2+2S^2+2R+2S+1,\quad 2R^2+2S^2+2R+2S+1,\quad 4R^2-4S^2+4R-4S$

$H=2\left(2R+1\right)\left(2S+1\right)\left(R^2-S^2+R-S\right)$

A few numerical examples are given in the Table 3.6 below:

Table: 3.6 Numerical examples

R	S	Heron Triple	Area	Perimeter
2	1	(17, 17, 16)	120	50
3	2	(37, 37, 24)	420	98
5	3	(85, 85, 72)	2772	242
4	2	(53, 53, 56)	756	162

It is observed that 2*Perimeter is a perfect square

2.3.2. Pattern: 2

Consider the sides of integer isosceles heron triangle to be a, a, b^2. Employing the Heron's formula for area H, we have

$$H=\frac{b^2}{4}\sqrt{4a^2-b^4} \tag{3.12}$$

The choices

$$a=2A,\quad b=2B \tag{3.13}$$

in (3.12) lead to

$$H=4B^2\sqrt{A^2-B^4} \tag{3.14}$$

To eliminate the square-root on the R.H.S of (3.14), assume

$$A^2-B^4=S^2 \tag{3.15}$$

which is in the form of well-known Pythagorean equation and is satisfied by

$$S=\alpha^2-\beta^2\ ,\ B^2=2\alpha\beta\ ,\ A=\alpha^2+\beta^2\ ,\ \alpha>\beta>0 \tag{3.16}$$

Let $\alpha=2^{2t-1}\beta^{2s-1},\quad t,s\geq 1$ (3.17)

Substituting (3.17) in (3.16), we have

$$S=2^{4t-2}\beta^{4s-2}-\beta^2$$

$$B=2^t\beta^s$$

$$A=2^{4t-2}\beta^{4s-2}+\beta^2$$

In view of (3.13), we have

$$a=2^{4t-1}\beta^{4s-2}+2\beta^2$$

$$b=2^{t+1}\beta^s$$

Hence, the corresponding sides and area H of the integer isosceles heron triangle are given by

Sides: $2^{4t-1}\beta^{4s-2}+2\beta^2\ ,\quad 2^{4t-1}\beta^{4s-2}+2\beta^2\ ,\quad 2^{2t+2}\beta^{2s}$

$H=2^{2t+2}\beta^{2s}\left(2^{4t-2}\beta^{4s-2}-\beta^2\right)$

A few numerical examples are given in the Table 3.7 below:

Table: 3.7 Numerical examples

t	s	β	Heron Triple	Area	Perimeter
2	1	2	(520, 520, 256)	64512	1296
1	2	3	(5850, 5850, 1296)	3767472	12996
3	2	2	(131080, 131080, 4096)	16760832	266256
3	1	4	(32800, 32800, 4096)	67043328	69696

Note that 6*Perimeter is a Nasty number.
Instead of (3.16), equation (3.15) is also satisfied by

$$\left.\begin{aligned} A &= \alpha^2+\beta^2 \\ S &= 2\alpha\beta \end{aligned}\right\} \quad (3.18)$$

$$B^2=\alpha^2-\beta^2\ , \quad \alpha>\beta>0 \quad (3.19)$$

Here equation (3.19) is again in the form of well-known Pythagorean equation and is satisfied for the following two choices:

i) $\alpha = p^2+q^2$, $\beta=2pq$, $B=p^2-q^2$, $p>q>0$ (3.20)

ii) $\alpha = p^2+q^2$, $\beta=p^2-q^2$, $B=2pq$, $p>q>0$ (3.21)

Consider choice (i). The corresponding sides and area H of the integer isosceles heron triangle are given by

Sides: $2(p^2+q^2)^2+8p^2q^2$, $2(p^2+q^2)^2+8p^2q^2$, $4(p^2-q^2)^2$

$$H=16pq\left(p^2+q^2\right)\left(p^2-q^2\right)^2$$

A few numerical examples are given in the Table 3.8 below:

Table: 3.8 Numerical examples

p	q	Heron Triple	Area	Perimeter
2	1	(82, 82, 36)	1440	200
3	2	(626, 626, 100)	31200	1352
5	3	(4112, 4112, 1024)	2088960	9248
4	2	(1312, 1312, 576)	368640	3200

It is observed that 3*Perimeter is a Nasty number.
For choice (ii), the corresponding sides and area H of the integer isosceles heron triangle are given by

Sides: $4(p^4+q^4)$, $4(p^4+q^4)$, $16p^2q^2$

$$H=32p^2q^2\left(p^4-q^4\right)$$

A few numerical examples are given in the Table 3.9 below:

Table: 3.9 Numerical examples

p	q	Heron Triple	Area	Perimeter
2	1	(68, 68, 64)	1920	200
3	2	(388, 388, 576)	74880	1352
5	3	(2824, 2824, 3600)	3916800	9248
4	2	(1088, 1088, 1024)	491520	3200

It is to be noted that 3*Perimeter is a Nasty number.
Also, Equation (3.15) is represented as the system of double equations as shown below:

System: 1 $A+S=B^4$, $A-S=1$

System: 2 $A+S=B^3$, $A-S=B$

Consider System: 1.The corresponding sides and area H of the integer isosceles heron triangle are given by

Sides: $16k^4+32k^3+24k^2+8k+2$, $16k^4+32k^3+24k^2+8k+2$, $4\left(4k^2+4k+1\right)$

$$H=4\left(4k^2+4k+1\right)\left(8k^4+16k^3+12k^2+4k\right)$$

A few numerical examples are given in the Table 3.10 below:

Table: 3.10 Numerical examples

k	Heron Triple	Area	Perimeter
2	(626, 626, 100)	31200	1352
3	(2402, 2402, 196)	235200	5000
5	(14642, 14642, 484)	3542880	29768
4	(6562, 6562, 324)	1062720	13448

Note that 2*Perimeter is a perfect square

For System: 2, the corresponding sides and area H of the integer isosceles heron triangle are given by

Sides: $B\left(B^2+1\right),\quad B\left(B^2+1\right),\quad 4B^2\;,\quad B>1$

$H = 2B^3\left(B^2-1\right)$

A few numerical examples are given in the Table 3.11 below:

Table: 3.11 Numerical examples

B	Heron Triple	Area	Perimeter
2	(10, 10, 16)	48	36
3	(30, 30, 36)	432	96
5	(130, 130, 100)	6000	360
2	(68, 68, 64)	1920	200

It is observed that 6*Perimeter is a Nasty number.

2.3.3. Pattern: 3

Let the sides of integer isosceles heron triangle be a, a, b^2+c^2. Employing the Heron's formula for area H, we have

$$H = \frac{b^2+c^2}{4}\sqrt{4a^2-\left(b^2+c^2\right)^2} \tag{3.22}$$

Let $x^2 = 4a^2-\left(b^2+c^2\right)^2$ (3.23)

which is in the form of well-known Pythagorean equation and is satisfied by

$$x = 2rs \tag{3.24}$$

$$b^2+c^2 = r^2-s^2 \tag{3.25}$$

$$2a = r^2+s^2\;,\quad r>s>0 \tag{3.26}$$

Here, equation (3.25) is in the form of space Pythagorean equation and is satisfied by

$$b = 2AB\;,\; c = 2AC\;,\; s = A^2-B^2-C^2\;,\; r = A^2+B^2+C^2 \tag{3.27}$$

In view of (3.24), (3.26), we have

$$a = A^4+\left(B^2+C^2\right)^2$$

$$x = A^4-\left(B^2+C^2\right)^2$$

Then, the corresponding sides and area H of the integer isosceles heron triangle are given by

Sides: $A^4+\left(B^2+C^2\right)^2\;,\quad A^4+\left(B^2+C^2\right)^2\;,\quad 4A^2\left(B^2+C^2\right)$

$H = 2\,A^2\left(B^2+C^2\right)\left(A^4-\left(B^2+C^2\right)^2\right),\quad A^2>B^2+C^2$

A few numerical examples are given in the Table 3.12 below:

Table: 3.12 Numerical examples

A	B	C	Heron Triple	Area	Perimeter
3	2	1	(106, 106, 180)	5040	392
4	2	3	(425, 425, 832)	36192	1682
5	3	2	(794, 794, 1300)	296400	2888
4	1	2	(281, 281, 320)	36960	882

It is observed that 3* Perimeter is a Nasty number.

2.3.4 Pattern: 4

Consider the sides of integer isosceles heron triangle to be a, a, b^3. Employing the Heron's formula for area H, we have

$$H = \frac{b^3}{4}\sqrt{4a^2 - b^6} \tag{3.28}$$

The choices

$$a = 4A, \quad b = 2B \tag{3.29}$$

in (3.28) lead to

$$H = 16B^3\sqrt{A^2 - B^6} \tag{3.30}$$

To eliminate the square-root on the R.H.S of (3.30), assume

$$X^2 = A^2 - B^6 \tag{3.31}$$

which is in the form of well-known Pythagorean equation and is satisfied by

$$X = p^2 - q^2, \; B^3 = 2pq, \; A = p^2 + q^2, \; p > q > 0 \tag{3.32}$$

Assume $p = 4\alpha^3 q^2$

In view of (3.32), we have

$$X = 16\alpha^6 q^4 - q^2, \quad B = 2\alpha q, \quad A = 16\alpha^6 q^4 + q^2$$

Then, the corresponding sides and area H of the integer isosceles heron triangle are given by

Sides: $64\alpha^6 q^4 + 4q^2, \quad 64\alpha^6 q^4 + 4q^2, \quad 64\alpha^3 q^3$

$$H = 128\alpha^3 q^3 \left(16\alpha^6 q^4 - q^2\right)$$

A few numerical examples are given in the Table 3.13 below:

Table: 3.13 Numerical examples

α	q	Heron Triple	Area	Perimeter
1	2	(1040, 1040, 8)	258048	2088
2	3	(331812, 331812, 24)	2292986880	663648
3	2	(746512, 746512, 24)	5159669760	1493048
2	2	(65552, 65552, 4096)	134184960	135200

Instead of (3.32), equation (3.31) is also satisfied by

$$X = 2pq \tag{3.33}$$

$$B^3 = p^2 - q^2 \tag{3.34}$$

$$A = p^2 + q^2, \quad p > q > 0 \tag{3.35}$$

One may write equation (3.34) as the system of double equations as shown below:

System: 1 $\quad p + q = B^2, \quad p - q = B$

System: 2 $\quad p + q = B^3, \quad p - q = 1$

Consider System: 1.The corresponding sides and area H of the integer isosceles heron triangle are given by

Sides: $2\left(B^4 + B^2\right), \quad 2\left(B^4 + B^2\right), \quad 8B^3$

$$H = 8B^3 \left(B^4 - B^2\right)$$

A few numerical examples are given in the Table 3.14 below:

Table: 3.14 Numerical examples

B	Heron Triple	Area	Perimeter
2	(40, 40, 64)	768	144
3	(180, 180, 216)	15552	576
5	(1300, 1300, 1000)	600000	3600
4	(544, 544, 512)	122880	1600

Note that 6*Perimeter is a Nasty number.

For System: 2, the corresponding sides and area H of the integer isosceles heron triangle are given by

Sides: $4\left(\left(4k^3 + 6k^2 + 3k + 1\right)^2 + \left(4k^3 + 6k^2 + 3k\right)\right), \quad 4\left(\left(4k^3 + 6k^2 + 3k + 1\right)^2 + \left(4k^3 + 6k^2 + 3k\right)\right), \quad 8(2k+1)^3$

$$H = 32\,(2k+1)^3 \left(4k^3 + 6k^2 + 3k + 1\right)\left(4k^3 + 6k^2 + 3k\right)$$

A few numerical examples are given in the Table 3.15 below:

Table: 3.15 Numerical examples

k	Heron Triple	Area	Perimeter
2	(4031, 4031, 1000)	15624000	9062
3	(29755, 29755, 2744)	322826112	62254
5	(444221, 444221, 10648)	18863570880	899090
4	(133589, 133589, 5832)	3099358080	273010

It is worth to note that, equation (3.31) is also represented as the system of double equations as shown below:

System: 1 $A+X=B^6$, $A-X=1$

System: 2 $A+X=B^5$, $A-X=B$

System: 3 $A+X=B^4$, $A-X=B^2$

Consider System: 1.The corresponding sides and area H of the integer isosceles heron triangle are given by

Sides:

$$128k^6+384k^5+480k^4+320k^3+120k^2+24k+4\,,\quad 128k^6+384k^5+480k^4+320k^3+120k^2+24k+4\,,\quad 8(2k+1)^3$$

$$H=16(2k+1)^3\left(32k^6+96k^5+120k^4+80k^3+30k^2+6k\right)$$

A few numerical examples are given in the Table 3.16 below:

Table: 3.16 Numerical examples

k	Heron Triple	Area	Perimeter
2	(31252, 31252, 1000)	15624000	63504
3	(235300, 235300, 2744)	322826112	473344
5	(3543124, 3543124, 10648)	18863570880	7096896
4	(1062884, 1062884, 5832)	3099358080	2131600

It is observed that Perimeter is a perfect square.

For System: 2, the corresponding sides and area H of the integer isosceles heron triangle are given by

Sides: $2B\left(B^4+1\right),\quad 2B\left(B^4+1\right),\quad 8B^3$

$$H=8B^4\left(B^4-1\right)$$

A few numerical examples are given in the Table 3.17 below:

Table: 3.17 Numerical examples

B	Heron Triple	Area	Perimeter
2	(68, 68, 64)	1920	200
3	(492, 492, 216)	51840	1200
5	(6260, 6260, 1000)	3120000	13520
4	(2056, 2056, 512)	522240	4624

Consider System: 3.The corresponding sides and area of the integer isosceles heron triangle are given by

Sides: $2B^2\left(B^2+1\right),\quad 2B^2\left(B^2+1\right),\quad 8B^3$

$$H=8B^5\left(B^2-1\right)$$

A few numerical examples are given in the Table 3.18 below:

Table: 3.18 Numerical examples

B	Heron Triple	Area	Perimeter
2	(40, 40, 64)	768	144
3	(180, 180, 216)	15552	576
5	(1300, 1300, 1000)	600000	3600
4	(544, 544, 512)	122880	1600

It is to be noted that 6*Perimeter is a Nasty number.

2.3.5. Pattern: 5

Let the sides of integer isosceles heron triangle be a, a, b^2-c^2 . Employing the Heron's formula for area H, we have

$$H=\frac{b^2-c^2}{4}\sqrt{4a^2-\left(b^2-c^2\right)^2} \tag{3.36}$$

Let $x^2 = 4a^2 - \left(b^2 - c^2\right)^2$ (3.37)

which is in the form of well-known Pythagorean equation and is satisfied by

$$x = 2rs \quad (3.38)$$

$$b^2 - c^2 = r^2 - s^2 \quad (3.39)$$

$$2a = r^2 + s^2\,, \quad r > s > 0 \quad (3.40)$$

Here equation (3.39) represents the numerical relation for R_2 numbers. Choosing suitable values for r and s and using (3.40), the value of a is obtained and the corresponding area is an integer.
A few numerical examples are given in the Table 3.19 below:

Table: 3.19 Numerical examples

Heron Triple	Area	Perimeter
(20, 20, 32)	192	72
(73, 73, 96)	2640	242
(601, 601, 480)	132240	1682
(1546, 1546, 780)	583440	3872
(3361, 3361, 3360)	4890480	10082
(6052, 6052, 12096)	1330560	24200
(114145, 114145, 106272)	5367958128	334562

It is observed that 3* Perimeter is a Nasty number.

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