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Non-Homogeneous Bi-Quadratic Equation With Three Unknowns $x^2+3xy+y^2=z^4$

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Abstract: *We obtain infinitely many non-zero integer triples (x, y, z) satisfying the non-homogeneous bi-quadratic equation with three unknowns $x^2+3xy+y^2=z^4$. Various interesting properties among the values of x, y, z are presented.*

Keywords: *Ternary bi-quadratic, Integer solutions.*

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I. Introduction

The theory of Diophantine equations offers a rich variety of fascinating problems. There is a great interest for mathematicians since antiquity in homogeneous and non-homogeneous bi-quadratic Diophantine equations [1-4]. In this context, one may refer [5-10] for varieties of problems on the bi-quadratic Diophantine equations with three variables. In [11-15], bi-quadratic equation with four unknowns are studied for their integral solutions. This communication concerns with yet another interesting ternary bi-quadratic equation given by $x^2+3xy+y^2=z^4$ and is analysed for its non-zero distinct integer solutions. Also, a few interesting relations between the solutions are presented.

II. Method of analysis

The ternary bi-quadratic Diophantine equation to be solved for its non-zero distinct integral solutions is given by

$$x^2+3xy+y^2=z^4 \tag{1}$$

Introducing the linear transformations

$$x=u+v\ ,\ y=u-v \tag{2}$$

in (1) leads to

$$5u^2-v^2=z^4 \tag{3}$$

We present below different methods of solving (3) and thus obtain different patterns of integral solutions to (1).

PATTERN: 1

One may write (3) as

$$5u^2=v^2+(z^2)^2 \tag{4}$$

Assume $u=u(a,b)=a^2+b^2$ (5)

Write 5 as

$$5=(2+i)(2-i) \tag{6}$$

Substituting (5) and (6) in (4) and employing the method of factorization, define

$$v+iz^2=(2+i)(a+ib)^2$$

Equating real and imaginary parts, we have

$$v=2(a^2-b^2)-2ab \tag{7}$$

$$z^2=a^2-b^2+4ab \tag{8}$$

The solution to (8) is

$$\left.\begin{aligned} a &= 5p^2+q^2-4pq \\ b &= 2pq \end{aligned}\right\} \qquad (9)$$

$$z = 5p^2 - q^2 \qquad (10)$$

Substituting (9) in (5) and (7) , we get

$$u = 25p^4 + q^4 + 30p^2q^2 - 8pq^3 - 40p^3q$$

$$v = 50p^4 + 2q^4 + 60p^2q^2 - 20pq^3 - 100p^3q$$

In view of (2), we have

$$\left.\begin{aligned} x &= 75p^4 + 3q^4 + 90p^2q^2 - 28pq^3 - 140p^3q \\ y &= -25p^4 - q^4 - 30p^2q^2 + 12pq^3 + 60p^3q \end{aligned}\right\} \qquad (11)$$

Thus (10) and (11) represents non zero distinct integer solutions to (1).

Properties:

1. $3(t_{4,q})^2 - x(1,q) - 14SO_q + 90\,\mathrm{Pr}_q + 75 \equiv 0 \pmod 2$
2. $x(1,q) + 3y(1,q) - 12OH_q \equiv 0 \pmod 2$
3. $x(q,q) + 3y(q,q) - 48(t_{4,q}) = 0$

PATTERN: 2

Rewrite (8) as

$$(a+2b)^2 - 5b^2 = z^2$$

$$(a+2b)^2 = 5b^2 + z^2$$

$$5b^2 + z^2 = (a+2b)^2 * 1 \qquad (12)$$

Assume $a + 2b = \alpha^2 + 5\beta^2$ (13)

Write 1 as

$$1 = \frac{(2+i\sqrt{5})(2-i\sqrt{5})}{9} \qquad (14)$$

Substituting (13) & (14) in (12) and employing the method of factorization, define

$$z + i\sqrt{5}b = \frac{1}{3}(2+i\sqrt{5})(\alpha + i\sqrt{5}\beta)^2$$

Equating the real and imaginary parts, we have

$$\left.\begin{aligned} z &= \frac{1}{3}(2\alpha^2 - 10\beta^2 - 10\alpha\beta) \\ b &= \frac{1}{3}(4\alpha\beta + \alpha^2 - 5\beta^2) \end{aligned}\right\} \qquad (15)$$

As our interest is on finding integer solutions, replacing α by $3P$ and β by $3Q$ in (13) and (15), we get

$$\left.\begin{aligned} a &= 3P^2 + 75Q^2 - 24PQ \\ b &= 3P^2 - 15Q^2 + 12PQ \end{aligned}\right\} \qquad (16)$$

$$z = 6P^2 - 30Q^2 - 30PQ \qquad (17)$$

Substituting (16) in (5) and (7), we get

$$u = 18P^4 + 5850Q^4 + 1080P^2Q^2 - 3960PQ^3 - 72P^3Q$$

$$v = -18P^4 + 13050Q^4 + 2160P^2Q^2 - 9000PQ^3 - 360P^3Q$$

In view of (2), we have

$$\left.\begin{aligned} x &= 18900Q^4 + 3240P^2Q^2 - 12960PQ^3 - 432P^3Q \\ y &= 36P^4 - 7200Q^4 - 10800P^2Q^2 + 5040PQ^3 + 288P^3Q \end{aligned}\right\} \qquad (18)$$

Thus (17) and (18) represents non zero distinct integer solutions to (1).

Properties:

1. $y(P,1)-36(t_{4,P})^2-144SO_P+1080\Pr_P+7200\equiv 0(\bmod 2)$
2. $z(1,Q)+30\Pr_Q\equiv 0(\bmod 2)$
3. $x(P,-P)-9612(t_{4,P})=0$

PATTERN: 3

In addition to (14), one may write 1 as

$$1=\frac{(2+i3\sqrt{5})(2-i3\sqrt{5})}{49} \tag{19}$$

Substituting (13) & (19) in (12) and employing the method of factorization define

$$z+i\sqrt{5}b=\frac{1}{7}(2+i3\sqrt{5})(\alpha+i\sqrt{5}\beta)^2$$

Equating the real and imaginary parts , we have

$$\left.\begin{aligned} z&=\frac{1}{7}(2\alpha^2-10\beta^2-30\alpha\beta)\\ b&=\frac{1}{7}(4\alpha\beta+3\alpha^2-15\beta^2)\end{aligned}\right\} \tag{20}$$

As our interest is on finding integer solutions, replacing α by $7P$ and β by $7Q$ in (13) and (20) , we get

$$\left.\begin{aligned} a&=7P^2+455Q^2-56PQ\\ b&=21P^2-105Q^2+28PQ\end{aligned}\right\} \tag{21}$$

$$z=14P^2-70Q^2-210PQ \tag{22}$$

Substituting (21) in (5) and (7) , we get

$$u=490P^4+218050Q^4+5880P^2Q^2-56840PQ^3+392P^3Q$$

$$v=-1078P^4+487550Q^4+11760P^2Q^2-127400PQ^3-1960P^3Q$$

In view of (2), we have

$$\left.\begin{aligned} x&=-588P^4+705600Q^4+17640P^2Q^2-184240PQ^3-1568P^3Q\\ y&=1568P^4-269500Q^4-5880P^2Q^2+70560PQ^3+2352P^3Q\end{aligned}\right\} \tag{23}$$

Thus (22) and (23) represents non zero distinct integer solutions to (1).

Properties:

1. $z(P,1)-t_{30,P}+70\equiv 0(\bmod 197)$
2. $y(P,1)-1568(t_{4,P})^2-1176OH_P+5880\Pr_P+269500\equiv 0(\bmod 2)$
3. $x(1,Q)-705600(t_{4,Q})^2+92120SO_Q-17640\Pr_Q+588\equiv 0(\bmod 2)$

NOTE:

It is worth to note that in addition to (14), one may write 1 as

$$1=\frac{(1+i4\sqrt{5})(1-i4\sqrt{5})}{81} \tag{24}$$

Following the procedure as presented above, the corresponding non-zero distinct integer solutions to (1) are given by

$$x=-1701P^4+1998675Q^4+7290P^2Q^2-199260PQ^3-972P^3Q$$

$$y=4455P^4-763425Q^4-2430P^2Q^2+76140PQ^3+2268P^3Q$$

$$z=9P^2-45Q^2-360PQ$$

Properties:

1. $z(P,1)-t_{20,P}+45\equiv 0(\bmod 2)$
2. $x(P,P)+y(P,P)-1121040(t_{4,P})^2=0$

3. $x(P,1)+1701(t_{4,P})^2+486SO_P-7290\,\mathrm{Pr}_P-1998675\equiv 0 \pmod 2$

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