

## A Transiently Chaotic flow for free convection of a couple stress fluids in a rotating porous layer

Kempe Gowda M<sup>1</sup>

<sup>1</sup>Department of Mathematics, Vemana Institute of Technology, Bengaluru, Karnataka-560034, India

---

**Abstract:** In this study, a dynamical system of Lorenz equation is discussed. The main aim of this paper is to demonstrate the possible convection regimes at supercritical values of the centrifugal Rayleigh number and evaluate the corresponding average rate of heat transfer for the problem of centrifugally driven convection of couple stress fluids in a porous layer placed at an offset distance from the axis of rotation. The solution to the non-linear dynamics is obtained by Galerkin technique. In a manner similar to the Lorenz formulation, single term Galerkin representations for the stream function is used. However, two terms are needed in the temperature expressions if some parts of the non-linearities are retained. The system of ordinary differential equations was solved by using Adomian's decomposition method. The critical Rayleigh number to chaos is obtained.

**Keywords**– Adomian's decomposition, Lorenz equations, Transiently Chaotic flows,

---

Date of Submission: 20-06-2018

Date of acceptance: 06-07-2018

---

### I. Introduction

In geophysical fluid dynamics involving large scale phenomena the effect of Coriolis force due to rotation plays an important role in the understanding of the mechanism of transfer of energy and momentum from one region to the other. The study of transport phenomena in rotating couple stress fluid in porous media is motivated by its practical applications in engineering and geophysics. The effect of rotation and of free convection as a result of the centrifugal body force is of particular interest from both the practical and theoretical points of view.

Engineering applications include among others, the food, chemical and materials processing industries, rotating machinery, chaos control, chaos synchronization and estimating the Lyapunov dimension of the chaotic attractors [3]. The problem of solidification of binary alloys includes also transport phenomena in rotating machinery. More specifically, packed bed mechanically agitated vessels are used in the food processing and chemical engineering industries in batch processes, the packed bed consists of solid particles or fibers of material which form the solid matrix while fluid flows through the pores, as the solid matrix rotates, due to the mechanical agitation, a rotating frame of reference is a necessity when investigating these flows. The role of the flow of fluid through these beds can vary from drying processes to extraction of soluble component from the solid particles. The molasses in centrifugal crystal separation processes in the sugar milling industry and the extraction of sodium alginate from kelp are just two examples of such processes.

Modelling of flow and heat transfer in porous media is also applied for the design of heat pipes using porous wicks and includes effects of boiling in unsaturated porous medium, surface tension driven flow with heat transfer and condensation in unsaturated porous media. When the heat pipes is used for cooling devices which are subject to rotation the corresponding centrifugal effect become relevant as well. A regenerator in a power plant can also be modelled by using the macro-level porous media approach. As the regenerator is typically subject to rotation the centrifugal effects on heat transfer are to be investigated.

Chaos can be observed in complex and nonlinear systems associated with saddle points [1]. The Critical Rayleigh number for the problem of a rotating layer adjacent to the axis of rotation as presented by Vadasz[9]. The stability map on the  $Ra_w$  to  $Ra_{w0}$  plane is presented showing the plane is divided between the

stable and unstable zones by the straight line  $\frac{Ra_{w,cr}}{7.81\pi^2} + \frac{Ra_{w0,cr}}{4\pi^2} = 1$ .

The results for the convective flow field were presented graphically by Vadasz[10] where it was concluded that the effect of the variation of the centrifugal acceleration within the porous layer is definitely felt when the box is close to the axis of rotation, corresponding to an eccentric shift of the convection cells towards the sidewall at  $x = 1$ . However the layer is located far away from the axis of rotation (e.g.,  $x = 50$ ) the convection cells are

concentric and symmetric with respect to  $x = \frac{1}{2}$ , as expected for a porous layer subject to gravity and heated from below.

Although the linear stability analysis is sufficient for obtaining the stability condition of the motionless solution and the corresponding Eigen functions describing qualitatively the convective flow it cannot provide information regarding the values of the convection amplitudes, nor regarding the average rate of heat transfer. To obtain this additional information, Vadasz and Olek [11] analysed and provided a solution to the nonlinear equations by using Adomian's decomposition method to solve a system of ordinary differential equations for the evolution of the amplitudes. The system of equations was obtained by using the first three relevant Galerkin modes for the stream function and the temperature while including the time derivative term in Darcy's equation in the form for homogeneous porous media. Keeping in view the practical applications in the field of biomechanics and growing importance of non-Newtonian fluids in modern technology and industries, the investigations on such fluids are desirable. Stokes [12] proposed and postulated the theory of couple-stress fluid. One of the mechanism of lubrication of Synovial joints, which has become the object of scientific research. A human joint is a dynamically loaded bearing which has articular cartilage as the bearing and Synovial fluid as the lubricant. When a fluid film is generated, squeeze film action is capable of providing considerable protection to the cartilage surface, the shoulder, knee, hip and ankle joint are the loaded-bearing synovial joints of the human body and these joints have a low-friction coefficient and negligible wear. Normal synovial fluid is clear of yellowish and is a viscous, non-Newtonian fluid. According to the theory of Stokes [12], Couple-stresses are found to appear in noticeable magnitude in fluids with very large molecules. The main goal of this paper is to introduce a couple stress fluids in the offset of rotation and discussion of the chaotic system.

## II. Formation of a problem

Let us consider a rotating fluid saturated porous layer subject to rotation is placed a dimensionless distance  $x_0$  from the centre of rotation as presented in figure 1. The offset distance is presented in a dimensionless form representing the ratio between the dimensional offset distance and the length of the porous layer in the form  $x_0 = \frac{x_0^*}{L_*}$ . Two systems of coordinates are presented in Fig. 1, the first  $(x', y', z')$  is linked to the axis of rotation and the second  $(x, y, z)$  are placed a horizontal distance apart from the first one, belongs to the porous layer coordinates. A positive temperature gradient in the  $x$ -direction is anticipated as a result of the imposed thermal boundary conditions. The significance of the variation of the centrifugal acceleration in the  $x$ -direction depends on the offset distance from the centre of rotation. The front aspect ratio of the layer is defined as  $a = \frac{H_*}{L_*}$  where  $H_*$  and  $L_*$  is the height and the length of the layer, respectively. The top aspect ratio is  $W = \frac{W_*}{L_*}$  where the width of the layer is  $W_*$ . The current analysis is restricted to a narrow layer, i.e.,  $W \ll 1$ ,

Free convection occurs as a result of the centrifugal body force while the gravity force is neglected. The only inertial effects considered are the centrifugal acceleration, as far as changes in density are concerned and the second is the time derivative term in Darcy's equation. Other than that, Darcy's law is assumed to govern the fluid flow (extended to include the aforementioned inertial terms), while the Boussinesq approximation is applied for the effect of density variations. As a narrow layer is considered, i.e.,  $W \ll 1$ , a Cartesian coordinate system can be used and the component of the centrifugal acceleration in the  $y$  direction can be neglected. Under these conditions the Governing equations, Continuity equation and Momentum equations from [9], [12] are given by

$$\nabla \cdot q = 0 \quad (1)$$

$$\left[ \frac{1}{\chi} \frac{\partial}{\partial t} + 1 \right] q = -\nabla p - [Ra_{w0} + Ra_w x] T \hat{e}_x + \left[ \frac{\mu_c}{\mu L_*^2} \right] \nabla^2 q \quad (2)$$

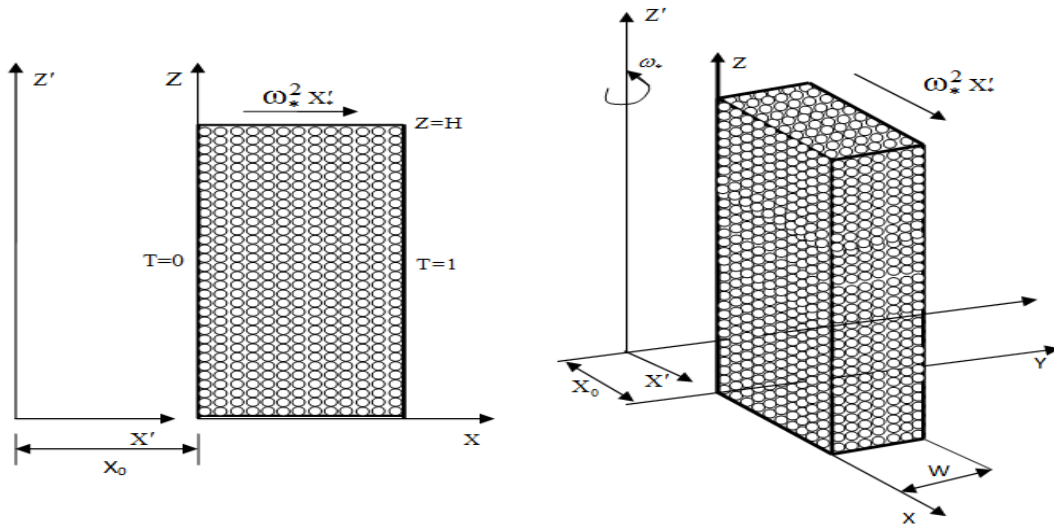
Where  $C = \frac{\mu_c}{\mu L_*^2}$ : Couple stress coefficient,  $\chi$  = a coefficient of the time derivative term in Darcy's equation

$$= \frac{\phi_{v_*} L_*^2}{\alpha_{e_*} \kappa_*}, \quad \phi = \text{Porosity}, \quad \nu_* = \text{fluids kinematic viscosity}, \quad \alpha_{e_*} = \text{effective thermal diffusivity}, \quad \kappa_* = \text{permeability of}$$

the porous domain,  $L_*$  = the length of the porous layer.

Temperature equation

$$\frac{\partial T}{\partial t} + q \cdot \nabla T = \nabla^2 T \quad (3)$$



**Figure1:** A rotating fluid saturated porous layer distant from the centre of rotation and subject to different temperatures at the sidewalls.

Equations (1) – (3) are presented in a dimensionless form as follows  $x = \frac{x_*}{L_*}$ ,  $y = \frac{y_*}{L_*}$ ,  $z = \frac{z_*}{L_*}$ ,  $t = \frac{t_* \alpha_e}{L_*^2}$ . We

observe two distinct centrifugal Rayleigh numbers.

(i)  $Ra_{w0} = \beta_* \Delta T_c w_*^2 x_0 L_* \kappa_* M_f / \alpha_e \nu_*$  represents the centrifugal Rayleigh number representing the contribution of the offset distance from the rotation axis to the centrifugal acceleration and.

(ii)  $Ra_w = \beta_* \Delta T_c w_*^2 L_*^2 \kappa_* M_f / \alpha_e \nu_*$  represents the contribution of the horizontal location within the porous layer to the centrifugal acceleration.

The reciprocal of the dimensionless offset distance from the axis of rotation  $\frac{1}{x_0}$  is introduced into equation (2)

$$\left[ \frac{1}{\chi} \frac{\partial}{\partial t} + 1 \right] q = -\Delta p - Ra_{w0} \left[ 1 + \frac{x}{x_0} \right] T \hat{e}_x + C \nabla^2 q \quad (4)$$

From equation (4) it is observed that when the porous layer is far away from the axis of rotation then  $x_0 \gg 1$

and the impact of the term  $\frac{x}{x_0}$  is not important, while for a layer close enough to the rotation axis  $x_0 \ll 1$  and

the involvement of the first term becomes immaterial. In the first case the only controlling parameter is  $Ra_{w0}$

while in latter case the only controlling parameter is  $Ra_w = \frac{Ra_{w0}}{x_0}$ .

As all the boundaries are rigid the solution must follow the impermeability conditions there, i.e.,  $q \cdot \hat{e}_n = 0$  on the boundaries, where  $\hat{e}_n$  is a unit vector normal to the boundary. The temperature boundary conditions are  $T = 0$  at  $x = 0$ ,  $T = 1$  at  $x = 1$  and  $\nabla T \cdot \hat{e}_n = 0$  on all other walls representing the insulation condition on these walls.

Let us introduce a stream function  $\psi(x, z, t)$  such that

$$u = \frac{\partial \psi}{\partial z} \quad \text{and} \quad w = -\frac{\partial \psi}{\partial x} \quad (5)$$

Then the continuity equation is identically satisfied and equation (4) becomes a partial differential equation as follows

$$\left[ \frac{1}{\chi} \frac{\partial}{\partial t} + 1 \right] \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} \right) = -[Ra_{w0} + Ra_w x] \frac{\partial T}{\partial x} + C \nabla^2 \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} \right) \quad (6)$$

Also the equation (3) gives

$$\frac{\partial T}{\partial t} + \frac{\partial \psi}{\partial z} \frac{\partial T}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial T}{\partial z} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \quad (7)$$

The set of partial differential equations (6) and (7) form a non-linear coupled system which together with the corresponding boundary conditions accepts a basic motionless conduction solution.

### III. Analysis

To obtain the solution to the non-linear coupled system of partial differential equations (6) and (7) the stream function and temperature are assumed to be of the form

$$\psi = A_{11} \sin(\pi x) \sin\left(\frac{\pi z}{H}\right) \quad (8)$$

$$T = x + B_{11} \sin(\pi x) \cos\left(\frac{\pi z}{H}\right) + B_{20} \sin(2\pi x) \quad (9)$$

This representation is equivalent to a Galerkin expansion of the solution in both  $x$  and  $z$  directions, truncated when  $i + j = 2$ , where  $i$  is the Galerkin summation index in the  $x$  direction and  $j$  is the Galerkin summation index in the  $z$  direction. Substituting (8) and (9) into the equation (6) we get

$$\left[ \frac{1}{\chi} \frac{\partial}{\partial t} + 1 \right] \pi A_{11} = -[Ra_{w0} + Ra_{wx}] \frac{B_{11}}{H} \left( 1 + \frac{1}{H^2} \right) - C^2 A_{11} \pi^3 \left( 1 + \frac{1}{H^2} \right)$$

On integrating them over the domain i.e.,  $\int_0^1 dx \int_0^H dz$

$$\frac{dA_{11}}{d\tau} = -\frac{\chi H}{\pi^2 \theta} \left[ \left( 1 + C \frac{\pi^2 \theta}{H} \right) A_{11} + \frac{\xi}{\theta \pi} B_{11} \right] \quad (10)$$

Substituting equations (8) and (9) into the equation (7) and comparing  $\sin(\pi x) \cos\left(\frac{\pi z}{H}\right)$  and  $2 \sin(\pi x) \cos(\pi x)$

coefficients we get

$$\frac{dB_{11}}{dt} + \frac{\pi A_{11}}{H} - \frac{2\pi^2 A_{11} B_{20}}{H} = -B_{11} \pi^2 \left( \frac{H^2 + 1}{H^2} \right)$$

$$\frac{dB_{20}}{dt} + \frac{A_{11} B_{11} \pi^2}{2H} = -4\pi^2 B_{20}$$

Integrating them over the domain i.e.,  $\int_0^1 dx \int_0^H dz$

$$\frac{dB_{11}}{d\tau} = -B_{11} - \frac{1}{\pi \theta} A_{11} + \frac{2}{\theta} A_{11} B_{20} \quad (11)$$

$$\frac{dB_{20}}{d\tau} = -\frac{1}{2\theta} A_{11} B_{11} - \frac{4H}{\theta} B_{20} \quad (12)$$

Where

$$\tau = \left( \frac{H^2 + 1}{H^2} \right) \pi^2 t; \quad \xi = Ra_{w0} + \frac{Ra_w}{2}; \quad \theta = \frac{H^2 + 1}{H} \quad (13)$$

The fixed or stationary points of the system of equations (10) (11) and (12) are obtained by setting all the time derivatives equal to zero and solving the resulting algebraic equations. They yield the following possible solutions.

$$A_{11} = B_{11} = B_{20} = 0 \quad (14)$$

$$A_{11} = \pm 2(H^2 + 1)^{1/2} \left[ \frac{\xi}{\pi^2 \theta^2} \left( 1 + C \pi^2 \frac{\theta}{H} \right)^{-1} - 1 \right]^{1/2} \quad (15)$$

$$B_{11} = \mp \frac{2H\pi\theta^2}{(H^2 + 1)^{1/2} \xi} \left[ 1 + C \pi^2 \frac{\theta}{H} \right] \left[ \frac{\xi}{\pi^2 \theta^2} \left( 1 + C \pi^2 \frac{\theta}{H} \right)^{-1} - 1 \right]^{1/2} \quad (16)$$

$$B_{20} = \frac{\theta^2 \pi}{2\xi} \left[ \frac{\xi}{\pi^2 \theta^2} - 1 - C \pi^2 \frac{\theta}{H} \right] \quad (17)$$

representing the steady convection solutions. It is convenient to introduce the following further notation

$$R = \frac{\xi}{\pi^2 \theta^2}, \quad \gamma = \frac{H^2}{H^2 + 1}, \quad \alpha = \frac{\chi \gamma}{\pi^2}, \quad \gamma^2 = \frac{H^2}{\theta^2} = \left( \frac{H^2}{H^2 + 1} \right)^2 \quad (18)$$

and rescale the amplitudes with respect to their convective fixed points in the form

$$A_{11} = \pm 2 \frac{H}{\sqrt{\gamma}} \left[ R \left( 1 + C \frac{\pi^2}{\gamma} \right)^{-1} - 1 \right]^{1/2} \quad \text{again } H = \theta_\gamma$$

$$X = \frac{-A_{11}}{2\theta \sqrt{\gamma \left[ R \left( 1 + \frac{C\pi^2}{\gamma} \right)^{-1} - 1 \right]}}$$

$$Y = \frac{\pi R B_{11}}{2 \sqrt{\gamma R \left( 1 + \frac{C\pi^2}{\gamma} \right) - \gamma \left( 1 + \frac{C\pi^2}{\gamma} \right)^2}} \quad (19)$$

$$Z = \frac{2\pi R}{\left[ R - 1 - \frac{C\pi^2}{\gamma} \right]}$$

to provide the following set of scaled equations with equivalent to equations (10) (11) and (12)

$$\dot{X} = \alpha_1 [-X + Y] \quad (20)$$

$$\dot{Y} = -Y + R \left( 1 + \frac{C\pi^2}{\gamma} \right)^{-1} X - \left[ R \left( 1 + \frac{C\pi^2}{\gamma} \right)^{-1} \right] XZ \quad (21)$$

$$\dot{Z} = 4\gamma [XY - Z] \quad (22)$$

Where  $X, Y, Z$  are the state variables  $\alpha_1, \gamma, R \left( 1 + \frac{C\pi^2}{\gamma} \right)^{-1}$  are the parameters and

$$\alpha_1 = \alpha \left\{ 1 + \frac{C\pi^2}{\gamma} \right\} = \frac{\chi H}{\pi^2 \theta} \left\{ 1 + \frac{C\pi^2}{\gamma} \right\}, \quad \gamma = \frac{H}{\theta}. \text{ Equations (20), (21) and (22) are equivalent to Lorenz equations$$

[5] with different coefficients. Since Lorenz equations were extensively analysed and solved for parameter values corresponding to gravity driven convection in pure fluids (i.e., non-porous domains) and even there the parameter values most frequently used correspond to  $\alpha = 10$  and  $\gamma = \frac{2}{3}$ . The Lorenz equations can be shown to be dissipative by using one of the Liapunov functions. Using the steady state techniques derived for fluid mechanics, we set  $\frac{\partial}{\partial t} = 0$  and let  $(X, Y, Z)$  with the steady solutions at the position vector  $(X_1, Y_1, Z_1)$ . Then equation (20) gives  $X = Y$ , equation (21) gives  $X = 0$  and  $Z = 0$ . Similarly, for  $X = 0$  equation (22) gives  $Z = 0$ . Therefore  $X_1 = Y_1 = Z_1 = 0$ , again for  $X = Y$  with equations (20) and (22)  $X^2 - Z = 0$  gives  $Z = X^2$  with equation (21) gives  $X_{2,3} = \pm 1, Y_{2,3} = \pm 1, Z_{2,3} = 1$ . So, the fixed points of the rescaled system are  $X_1 = Y_1 = Z_1 = 0$  corresponding to the motionless solution, and  $X_{2,3} = \pm 1, Y_{2,3} = \pm 1, Z_{2,3} = 1$  corresponding to the convection solution.

The next step is to perform a stability analysis of the stationary solutions in order to determine the nature of the dynamics about the fixed points. The system equations (20), (21) and (22) have the general form  $\dot{X} = f(X)$  and the equilibrium (stationary or fixed) points  $X_s$  are defined by  $f(X_s) = 0$ . The stability matrix is

established by evaluating the Jacobian  $\left( \frac{\partial f_i}{\partial x_j} \right)_{x_s}$  at the fixed point of interest  $X_s$ . The eigen values of the stability

matrix, evaluated by solving the zeros of the characteristic polynomial associated with the stability matrix, provide the stability conditions. A fixed point is stable if all eigen values corresponding to its stability matrix are negative [or in the case of complex eigenvalues they have negative real parts] and it is not stable if at least one eigen value becomes positive [or in the case of complex eigen values it has a positive real part] From equations (20), (21), (22) and linear steady we have

$$\begin{aligned} -\alpha_1 X + \alpha_1 Y &= 0 \\ RX - \left(1 + \frac{C\pi^2}{\gamma}\right) Y &= 0 \\ -4\gamma Z &= 0 \end{aligned} \quad (23)$$

The Jacobian matrix of these equations is  $J = \begin{pmatrix} -\alpha_1 & \alpha_1 & 0 \\ R & -\left(1 + \frac{C\pi^2}{\gamma}\right) & 0 \\ 0 & 0 & -4\gamma \end{pmatrix}$ . The characteristic equation of system

(23) which is obtained from  $|J - \lambda I| = 0$  evaluated on the equilibrium is

$$(4\gamma + \lambda) \left[ \alpha_1 \left( R - 1 - \frac{C\pi^2}{\gamma} \right) - \left( \alpha_1 + 1 + \frac{C\pi^2}{\gamma} \right) \lambda - \lambda^2 \right] = 0 \quad (24)$$

Thus the first eigen value  $\lambda_1 = -4\gamma$  is always negative as  $\gamma > 0$ . The other two eigen values

$$\lambda_{2,3} = \frac{\left(1 + \frac{C\pi^2}{\gamma}\right)}{2} \left[ -(\alpha_1 + 1) \pm \sqrt{(\alpha_1 + 1)^2 + 4\alpha_1 \left( \frac{R - 1 - \frac{C\pi^2}{\gamma}}{1 + \frac{C\pi^2}{\gamma}} \right)} \right].$$

It can be seen that the solutions stability is

fundamentally influenced by the value of R.  $\lambda_3$  is always negative and  $\lambda_2$  provides the stability condition for the motionless solution in the form  $R < 1 + \frac{C\pi^2}{\gamma}$ . Therefore the critical value of R, where the motionless solution

loses stability and the convection solution (expressed by the other two fixed points) takes over, is obtained as  $R_{cr} = R_{cr} = 1 + \frac{C\pi^2}{\gamma}$ . The significance of this result can be established by expressing it in terms of the original

parameters by using equations (3) and (13) i.e.,  $\frac{\xi}{\pi^2 \theta^2} = 1 + \frac{C\pi^2}{\gamma}$  or  $Ra_{w0} + \frac{Ra_w}{2} = \pi^2 \theta^2 + \frac{C\pi^4 \theta^3}{H}$  two limiting cases of interest are

(i)  $x_0 \rightarrow 0$  corresponding to  $RA_{w0} \rightarrow 0$  which yields  $Ra_{w,cr} = 2\pi^2 \theta^2 + \frac{2C\pi^4 \theta^3}{H}$  and

(ii)  $x_0 \gg 1$  (say  $x_0 \rightarrow \infty$ ) which yields  $Ra_{w0,cr} = \pi^2 \theta^2 + \frac{C\pi^4 \theta^3}{H}$

The value of  $\theta$  has to be consistent with the wavenumber at the stability threshold in order for the convection cells to fit in to the domain and satisfy the boundary conditions. For case (i) the value of  $\theta_{cr}$  was found by Vadasz [10] to be  $\theta_{cr} = 2.0003$  while for case (ii)  $\theta_{cr} = 2$ . Substituting these values yields for case (i)

$$Ra_{w,cr} = 8.0024\pi^2 + \frac{2C}{H} (8.0036)\pi^4 \text{ and for case (ii) } Ra_{w,cr} = 4.0012\pi^2 + \frac{C}{H} (8.0036)\pi^4$$

for the unsteady case the characteristic equation is given by

$$\lambda^3 + (\alpha_1 + 1 + 4\gamma)\lambda^2 + 4\gamma \left[ \alpha_1 + R \left(1 + \frac{C\pi^2}{\gamma}\right)^{-1} \right] \lambda + 8\gamma\alpha_1 \left[ R \left(1 + \frac{C\pi^2}{\gamma}\right)^{-1} - 1 \right] = 0 \quad (25)$$

The stability of the fixed points associated with the convection solutions  $(X_{2,3}, Y_{2,3}, Z_{2,3})$  is controlled by the following cubic equation for the eigen values,  $\lambda_i$  ( $i = 1, 2, 3$ ). The smallest eigen value  $\lambda_1$  is always real and negative at slightly supercritical values of R, therefore the convection fixed points are stable, i.e., simple nodes. As the value of R increases the convection fixed points move on the real axis towards the origin, the smallest

between the two chasing the one and reducing the distance between them. For  $\alpha = \frac{50}{\pi^2}$  and  $\gamma = \frac{(3\pi)^4 - 2^{10}}{2(3\pi)^4}$  and

$C = 0.5$  these roots become equal when  $R \cong 15.06$ . It is exactly at this point when these two roots become complex conjugate. However, they have still negative real parts, therefore the convection fixed points are stable, i.e., spiral nodes. As the value of R increases further, both the imaginary and real parts of these two complex

conjugate eigen values increases and on the complex plane, they cross the imaginary axis i.e., the product of the coefficients of  $\lambda^2$  and  $\lambda$  equals the constant term or their real part becomes non-negative at a value at a value of  $R$  given by

$$R = \frac{\alpha [3\gamma + 3C\pi^2 + \alpha_1\gamma + 4\gamma^2 + \alpha_1 C\pi^2 + 4C\gamma\pi^2]}{\gamma [\alpha_1 - 1 - 4\gamma]} \quad (26)$$

At this point the convection fixed points lose their stability and chaos sets in. Just prior to this happening, at the point when the complex eigen values cross the imaginary axis a Hopf bifurcation occurs, i.e., at  $R_{c_2}$  these eigen values are purely imaginary leading to a limit cycle (periodic solution). The evolution of the complex eigen values for  $\alpha = \frac{50}{\pi^2}$  and  $\gamma = \frac{(3\pi)^4 - 2^{10}}{2(3\pi)^4}$  and  $C=0.5$  the loss of stability of the convection fixed points is evaluated using equation (25) to be

$$R_{c_2} = \frac{\alpha \left(1 + \frac{C\pi^2}{\gamma}\right) \left[3C\pi^2 + 3\gamma + \alpha\gamma \left(1 + \frac{C\pi^2}{\gamma}\right) + 4\gamma^2 + \alpha C\pi^2 \left(1 + \frac{C\pi^2}{\gamma}\right) + 4C\gamma\pi^2\right]}{\gamma \left[\alpha \left(1 + \frac{C\pi^2}{\gamma}\right) - 1 - 4\gamma\right]}$$

i.e.,  $R_{c_2} = 868.187$

An interesting observation can be made by investing the behaviour of the value of  $R_{c_2}$  as  $\alpha_1$  becomes very big. When  $\alpha_1 \gg 1$  the time derivative term in Darcy's equation and consequently in equation (20) is very small and this fact provides the justification of neglecting this term. By taking the limit of  $R_{c_2}$  in equation (26) one obtains that  $R_{c_2} \rightarrow \alpha$  as  $\alpha \rightarrow \infty$ . Therefore although a much higher value for transition to chaos is required when the values of  $\alpha$  are very big this transition still exists, while neglecting the time derivative term reduces to two of course, alternatively one could use more modes in the Galerkin expansion thus still allowing this transition to occur. However at this level of truncation including the time derivative term is necessary in order to keep all possibilities of transitions in the model.

#### IV. Method of solution

Adomian's decomposition method[1,2] is applied to solve the system of equations (20),(21) and (22). The method provides in principle an analytical solution in the form of an infinite power series for each dependent variable and its excellent accuracy in solving non-linear equations was demonstrated by [7, 11] Consider the following general dynamical system of equations.

$$\frac{dX_i}{dt} = \sum_{j=1}^m b_{ij} X_j + \sum_{l=1}^m \sum_{j=1}^m \alpha_{ijl} X_j X_l, \quad \forall i = 1, 2, \dots, m \quad (27)$$

given the initial conditions  $X_i(0), i = 1, 2, \dots, m$ . It can be easily observed that the system of equations (20), (21) and (22) is just a particular case of equation (27). Olek [7, 11] used the decomposition method to solve a variety of non-linear problems, some of which have closed from analytical solutions and a comparison was provided between the results obtained via the decomposition method and either exact analytical or numerical results. The problem can be solved to the desired accuracy by including more terms in the computation of the series.

For the system of equation (3) the non-linear terms are of the rather simple  $X^2$  form, so that very simple symmetry rules for the decomposition polynomials can be used. If we denote  $L = \frac{d}{dt}$ , the formal solution of equation (27) may be presented in the form.

$$X_i(t) = X_i(0) + L^{-1} \left[ \sum_{j=1}^m b_{ij} X_j + \sum_{l=1}^m \sum_{j=1}^m \alpha_{ijl} X_j X_l \right] \quad \forall i = 1, 2, \dots, m \quad (28)$$

i.e.,  $X = X(0) + L^{-1} [-\alpha_1 X + \alpha_1 Y]$

$$Y = Y(0) + L^{-1} \left[ R \left( 1 + \frac{C\pi^2}{\gamma} \right)^{-1} X - Y - \left[ R \left( 1 + \frac{C\pi^2}{\gamma} \right)^{-1} - 1 \right] XZ \right]$$

$$Z = Z(0) + L^{-1} [-4\gamma Z + 4\gamma XY]$$

Where  $L^{-1} = \int_0^t [\cdot] dt$ . According to the decomposition method an expansion of the following form is assumed.

$$X_i(t) = \sum_{n=0}^{\infty} \tilde{X}_{in}, \quad \forall i = 1, 2, \dots, m \quad (29)$$

$$\text{i.e., } X = \sum_{n=0}^{\infty} \tilde{X}_n, \quad Y = \sum_{n=0}^{\infty} \tilde{Y}_n, \quad Z = \sum_{n=0}^{\infty} \tilde{Z}_n$$

substituting equation (29) into equation (28) yields after rearranging the products.

$$X_i(t) = X_i(0) + L^{-1} \left[ \sum_{j=1}^m b_{ij} \sum_{n=0}^{\infty} \tilde{X}_{jn} + \sum_{l=1}^m \sum_{j=1}^m \alpha_{ijl} \sum_{n=0}^{\infty} \sum_{k=0}^n \tilde{X}_{jk} \tilde{X}_{l(n-k)} \right] \quad \forall i = 1, 2, \dots, m \quad (30)$$

The solution is ensured by requiring

$$X_0 = X_{10} \quad Y_0 = X_{20} \quad Z_0 = X_{30} \quad (31)$$

$$\tilde{X}_{i1} = L^{-1} \left[ \sum_{j=1}^m b_{ij} \tilde{X}_{j0} + \sum_{l=1}^m \sum_{j=1}^m \alpha_{ijl} \sum_{k=0}^0 \tilde{X}_{jk} \tilde{X}_{l(0-k)} \right], \quad \forall i = 1, 2, \dots, m \quad (32)$$

$$\tilde{X}_{i2} = L^{-1} \left[ \sum_{j=1}^m b_{ij} \tilde{X}_{j1} + \sum_{l=1}^m \sum_{j=1}^m \alpha_{ijl} \sum_{k=0}^1 \tilde{X}_{jk} \tilde{X}_{l(1-k)} \right], \quad \forall i = 1, 2, \dots, m \quad (33)$$

$$\tilde{X}_{i3} = L^{-1} \left[ \sum_{j=1}^m b_{ij} \tilde{X}_{j2} + \sum_{l=1}^m \sum_{j=1}^m \alpha_{ijl} \sum_{k=0}^2 \tilde{X}_{jk} \tilde{X}_{l(2-k)} \right], \quad \forall i = 1, 2, \dots, m \quad (34)$$

Similar to equations (32), (33) and (34) we have

$$X_1 = L^{-1} [-\alpha_1 X_0 + \alpha_1 Y_0],$$

$$Y_1 = L^{-1} \left[ R \left( 1 + \frac{C\pi^2}{\gamma} \right)^{-1} X_0 - Y_0 - \left[ R \left( 1 + \frac{C\pi^2}{\gamma} \right)^{-1} - 1 \right] X_0 Z_0 \right],$$

$$Z_1 = L^{-1} [-4\gamma Z_0 + 4\gamma X_0 Y_0]$$

From equations (32) to (34) we have

$$X_1 = L^{-1} \left[ b_{11} X_0 + b_{12} Y_0 + b_{13} Z_0 + a_{111} X_0 X_0 + a_{121} Y_0 X_0 + a_{131} Z_0 X_0 + a_{112} X_0 Y_0 \right. \\ \left. + a_{122} Y_0 Y_0 + a_{132} Z_0 Y_0 + a_{113} X_0 Z_0 + a_{123} Y_0 Z_0 + a_{133} Z_0 Z_0 \right]$$

$$Y_1 = L^{-1} \left[ b_{21} X_0 + b_{22} Y_0 + b_{23} Z_0 + a_{211} X_0 X_0 + a_{221} Y_0 X_0 + a_{231} Z_0 X_0 + a_{212} X_0 Y_0 \right. \\ \left. + a_{222} Y_0 Y_0 + a_{232} Z_0 Y_0 + a_{213} X_0 Z_0 + a_{223} Y_0 Z_0 + a_{233} Z_0 Z_0 \right]$$

$$Z_1 = L^{-1} \left[ b_{31} X_0 + b_{32} Y_0 + b_{33} Z_0 + a_{311} X_0 X_0 + a_{321} Y_0 X_0 + a_{331} Z_0 X_0 + a_{312} X_0 Y_0 \right. \\ \left. + a_{322} Y_0 Y_0 + a_{332} Z_0 Y_0 + a_{313} X_0 Z_0 + a_{323} Y_0 Z_0 + a_{333} Z_0 Z_0 \right]$$

$$X_2 = L^{-1} \left[ b_{11} X_1 + b_{12} Y_1 + b_{13} Z_1 + a_{111} \{X_0 X_1 + X_1 X_0\} + a_{121} \{Y_0 X_1 + Y_1 X_0\} \right. \\ \left. + a_{131} \{Z_0 X_1 + Z_1 X_0\} + a_{112} \{X_1 Y_0 + X_0 Y_1\} + a_{122} \{Y_1 Y_0 + Y_0 Y_1\} \right. \\ \left. + a_{132} \{Z_1 Y_0 + Z_0 Y_1\} + a_{113} \{X_1 Z_0 + X_0 Z_1\} + a_{123} \{Y_1 Z_0 + Y_0 Z_1\} + a_{133} \{Z_1 Z_0 + Z_0 Z_1\} \right]$$

$$Y_2 = L^{-1} \left[ b_{21} X_1 + b_{22} Y_1 + b_{23} Z_1 + a_{211} \{X_0 X_1 + X_1 X_0\} + a_{221} \{Y_0 X_1 + Y_1 X_0\} \right. \\ \left. + a_{231} \{Z_0 X_1 + Z_1 X_0\} + a_{212} \{X_1 Y_0 + X_0 Y_1\} + a_{222} \{Y_1 Y_0 + Y_0 Y_1\} \right. \\ \left. + a_{232} \{Z_1 Y_0 + Z_0 Y_1\} + a_{213} \{X_1 Z_0 + X_0 Z_1\} + a_{223} \{Y_1 Z_0 + Y_0 Z_1\} + a_{233} \{Z_1 Z_0 + Z_0 Z_1\} \right]$$

$$Z_2 = L^{-1} \left[ b_{31} X_1 + b_{32} Y_1 + b_{33} Z_1 + a_{311} \{X_0 X_1 + X_1 X_0\} + a_{321} \{Y_0 X_1 + Y_1 X_0\} \right. \\ \left. + a_{331} \{Z_0 X_1 + Z_1 X_0\} + a_{312} \{X_1 Y_0 + X_0 Y_1\} + a_{322} \{Y_1 Y_0 + Y_0 Y_1\} \right. \\ \left. + a_{332} \{Z_1 Y_0 + Z_0 Y_1\} + a_{313} \{X_1 Z_0 + X_0 Z_1\} + a_{323} \{Y_1 Z_0 + Y_0 Z_1\} + a_{333} \{Z_1 Z_0 + Z_0 Z_1\} \right]$$



where  $b_{11} = -\alpha_1 = -\alpha \left(1 + \frac{C\pi^2}{\gamma}\right) = -b_{12}$ ,  $b_{21} = R \left(1 + \frac{C\pi^2}{\gamma}\right)^{-1}$ ,  $b_{22} = -1$ ,  $b_{33} = -4\gamma$

Therefore, the following solution is obtained

$$X_i(t) = \sum_{n=0}^{\infty} C_{in} \frac{t^n}{n!}, \quad \forall i = 1, 2, \dots, m \quad (35)$$

where  $C_{i0} = X_i(0) \quad \forall i = 1, 2, \dots, m$  (36)

And the general term for  $n \geq 1$  is defined through the following recurrence relation.

$$C_{in} = \sum_{j=1}^m b_{ij} C_{j(n-1)} + (n-1)! \sum_{l=1}^m \sum_{j=1}^m \sum_{k=0}^{n-1} a_{ijl} \frac{C_{jk}}{k!} \frac{C_{l(n-k-1)}}{(n-k-1)!}, \quad \forall i = 1, 2, \dots, m \quad (37)$$

The decomposition method does not guarantee on its own existence and uniqueness of the solution. In fact, it can be safely applied when a fixed point theorem holds. A theorem proved by Repaci [8] indicates that there is no point in looking for solutions universally in time. On the other hand, the decomposition method can be used as algorithm for the approximation of the dynamical response in a sequence of time intervals. In this we observed that the solution obtained by the decomposition method converges to a unique solution as the number of terms in the series becomes infinite and the approximation in each interval of time is continuous with desired level of accuracy corresponding to the desire number of terms.

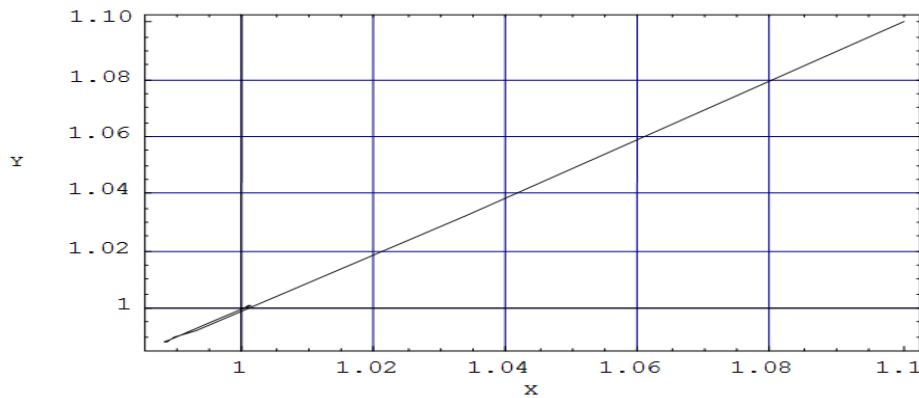
For  $m = 3$  in equations (20), (21) and (22) the above said computation of solution is adopted with the following

substitutions  $b_{11} = -\alpha_1 = -\alpha \left(1 + \frac{C\pi^2}{\gamma}\right)$ ,  $b_{12} = \alpha_1 = \alpha \left(1 + \frac{C\pi^2}{\gamma}\right)$ ,  $b_{21} = R \left(1 + \frac{C\pi^2}{\gamma}\right)^{-1}$ ,  $b_{22} = -1$ ,  $b_{33} = -4\gamma$ ,

$a_{213} = -\left[ R \left(1 + \frac{C\pi^2}{\gamma}\right)^{-1} - 1 \right]$ ,  $a_{312} = 4\gamma$  the remaining coefficients all are identically zero.

### V. Results and Discussions

The Adomian's decomposition method of solution presented in the previous section was applied to obtain the sets of results for different critical values of R. All solutions were obtained at origin.

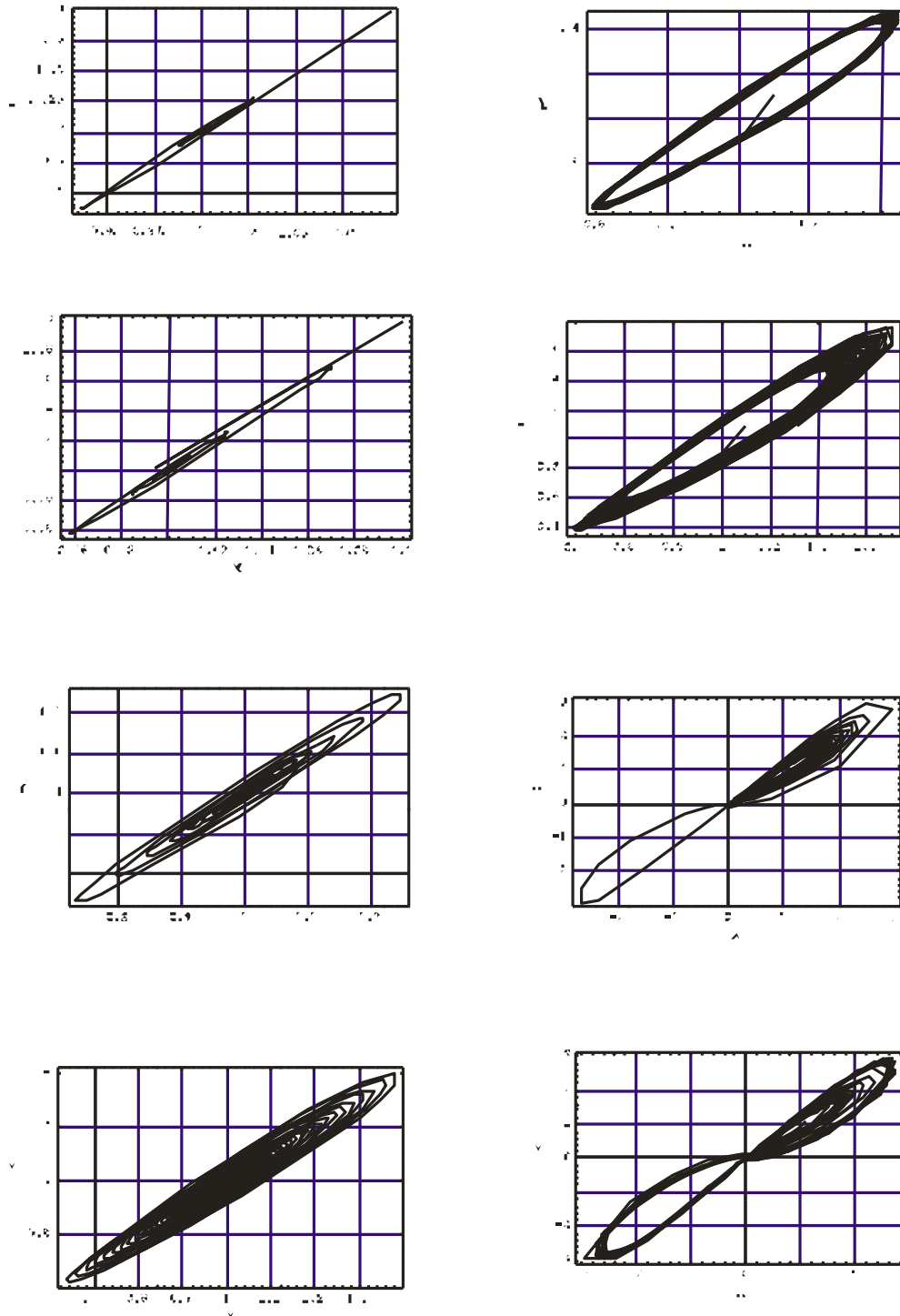


**Figure 2:** The projection of the solution data onto X-Y plane for

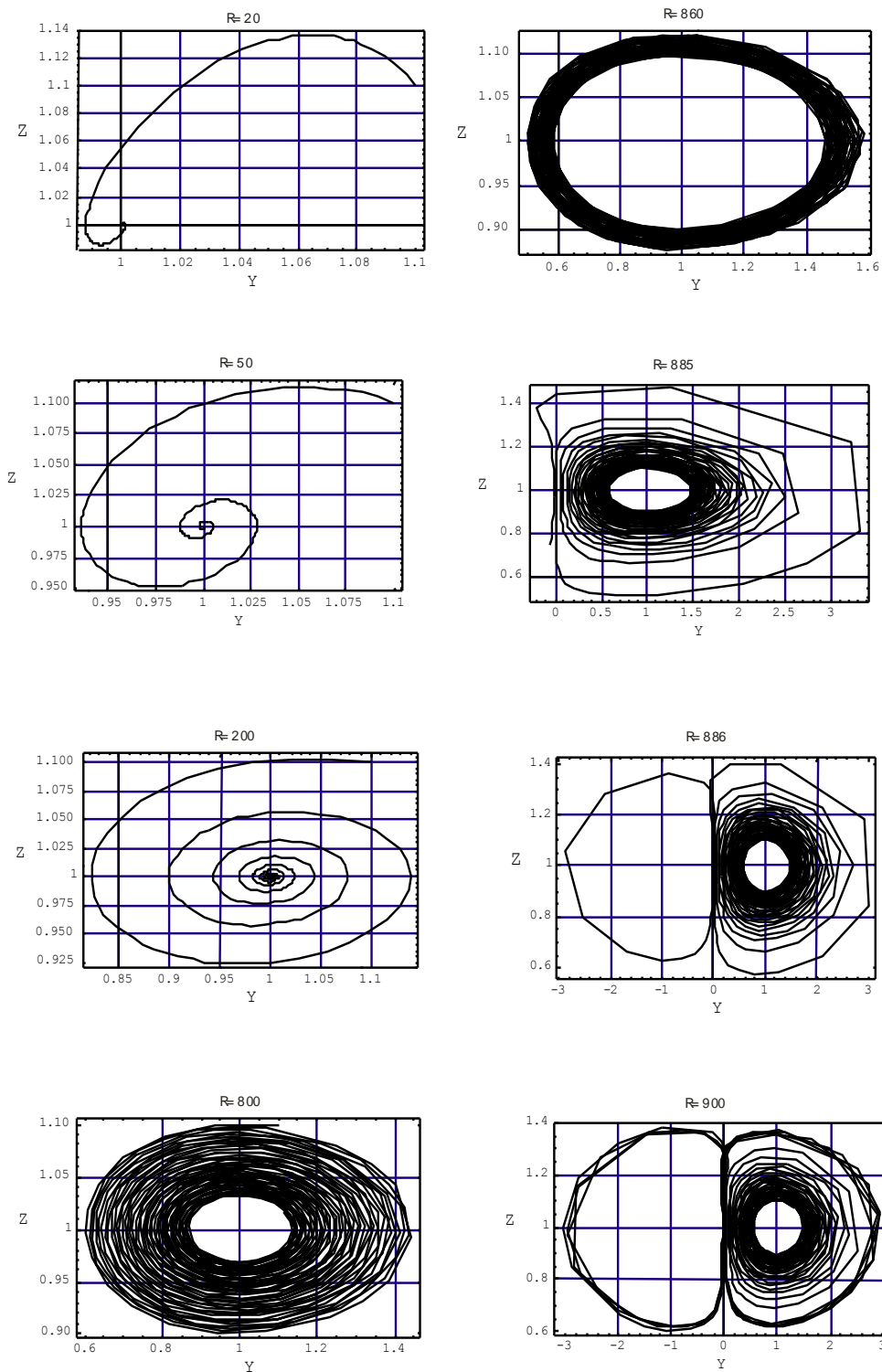
$$\alpha = \left(1 - \frac{C\pi^2}{\gamma}\right), \quad \gamma = \frac{(3\pi)^4 - 2^{10}}{2(3\pi)^4}, \quad C = 0.5, \quad R = 20$$

### VI. Conclusions

A nonlinear centrifugally driven free convection in a rotating couple stress fluid porous layer was solved for a truncated Galerkin approximation. It was demonstrated that the set of non-linear ordinary differential equations reduces to Lorenz equations with different coefficients. The Analytical method Adomian decomposition method was used to solve the equations. The results were demonstrated with different transitions. That is from steady convection to a non-periodic regime through Hopf bifurcation, and a further transition from chaos to periodic convection at significantly higher values of the centrifugal Rayleigh number. The first transition is confirmed by a stability analysis of the steady convection. These transitions show a marked impact on the average heat transfer, all the reservations which are applicable to Lorenz model are applicable here as well and developing a model which includes more Galerkin terms is indeed required for improving the accuracy regarding the different transitions and in particular at the higher values of the centrifugal Rayleigh number.



**Figure 3:** The projection of the solution data onto X-Y plane for  $\alpha = \left(1 - \frac{C\pi^2}{\gamma}\right)$ ,  $\gamma = \frac{(3\pi)^4 - 2^{10}}{2(3\pi)^4}$ ,  $C = 0.5$ , with different R values



**Figure 4:** The projection of the solution data onto Z-Y plane for  $\alpha = \left(1 - \frac{C\pi^2}{\gamma}\right)$ ,  $\gamma = \frac{(3\pi)^4 - 2^{10}}{2(3\pi)^4}$ ,  $C = 0.5$ , with different R values

### References

- [1]. Adomian G., A review of the decomposition method in applied mathematics. J. Math. Annual. Appl., 1988, 135, 501-544
- [2]. Abdul-Majid Wazwaz., a reliable modification of Adomian decomposition method. Applied Mathematics and Computation, 1999, 102, 77-85.
- [3]. F Zhang, X Liao, G Zhang and C Mu., Complex dynamics of a new 3D Lorenz-type autonomous chaotic system. Pramana-J. Phys. 2017, 89(6), 35-42

- [4]. Long-JyeSheu, Lap-Mou Tom, Juhn-horngchen, Hsien-kengchen, Kuang-Tai lin, Yuan kang., Choatic convection of viscoelastic fluids in porous media. *Chaos, Solitions and Fractals* 2008, 37, 113-124.
- [5]. Lorenz, E. N., Deterministi non-periodic flows. *J. Atmos. Sci.* 1963, 20, 130-141.
- [6]. Malashetty, M. S. and PremilaKollur, The onset of double diffusive convection in a couple stress fluid saturated anisotropic porous layer. *Transp. Porous Med.* 2010.
- [7]. Olek, S., An accurate solution to the multispecies Lotka-Volterra equations *SIAM Review*, 1994, 36, 480-488.
- [8]. Repaci, A., Non-Linear dynamical systems; on the accuracy of Adomian's decomposition method. *Appl. Math. Lett.*, 1990, 3, 35-39.
- [9]. Vadasz, P., Stability of free convection in a narrow porous layer subject to rotation. *Int. Comm. Heat Mass Transfer* 1994, 21(6), 881-890.
- [10]. Vadasz, P., Stability of free convection in a rotating porous layer distance from the axis rotation. *Transport in Porous Media*
- [11]. Vadasz, P. and Olek, S., Transitions and Chaos for free Convection in a rotating porous layer. *Int. J. Heat Mass Transfer* 1998, 41, 1417-1435.
- [12]. V. K. Stokes., Couple stresses in Fluids. *Phys. Fluids* 1966, 9, 1709-1715.

Kempe Gowda M" A Transiently Chaotic flow for free convection of a couple stress fluids in a rotating porous layer "International Journal of Engineering Science Invention (IJESI), vol. 07, no. 07, 2018, pp 51-62