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Fixed Point Theorems on Partial Metric Spaces Using Meir-Keeler Type Contractions

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Abstract: *Inthis paper we prove a common fixed point theorem in partial metric space for two pairs of weakly compatible self-mappings satisfying a generalized Meir –Keeler type contractive conditions. The presented theorem extends several well-known results in literature.*
Keywords:*Fixed Point, Partial Metric Space, Meir-Keeler type Contractions*

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I. Introduction

Matthews [1] introduced the partial metric spacesin which the distance of a point in the self may not be zero. The main objective is to study denotational semantics of data flow networks. In fact, (complete) partial metric spaces constitute a suitable framework to model several distinguished examples of the theory of computation and also to model metric spaces via domain theory. Partial metric spaces have serious applications potentials in the research area of computer domains and semantics, (see for example, [2, 3, 4, 5]).

In **1994,** Matthews [1] generalized the Banach contraction principle to the class of complete partial metric spaces: a self-mappingTon a complete partial metric space (X, p) has a unique fixed point if there exists $0 \leq k < 1$ such that

$$p\,(Tx, Ty) \leq kp\,(x, y) \text{for all } x, y \in X.$$

Recently, many authors have focused on this subject and generalized some fixed point theorems from the class of metric spaces to the class of partial metric spaces (see e.g., [1-28,]).

Later on, S.J. O'Neill generalized Matthews' notion of partial metric, in order to establish connections between these structures and the topological aspects of domain theory. S.Oltra and O. Valero [24]in **2004** obtained following Banach fixed point theorem for complete partial metric spaces in the sense of O'Neill.

Let f be a mapping of a complete dualistic partial metric space (X, p) into itself such that there is a real number c with $0 \leq c < 1$ satisfying:

$$|p(f(x), f(y))| \leq c|p(x, y)|,$$

for all $x, y \in X$. Then f has a unique fixed point

Bouhadjera, H, and Djoudi, A. [29] proved in**2008** two common fixed point results of Meir and Keeler type for four weakly compatible mappings:

Let $(A, S)$and(B, T) be weakly compatible pairs of self mappings of a complete metric space(X, d) such that the following conditions hold:
(a) $AX \subseteq TX$ and$BX \subseteq SX$,
(b) Oneof AX, BX, SX or TX is closed,
(c) Foreach$\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon < M(x, y) < \varepsilon + \delta \Rightarrow d(Ax, By) \leq \varepsilon,$$

(c')$x, y \in X, M(x, y) > 0 \Rightarrow d(Ax, By) < M(x, y)$,
where $M(x, y) = max\{d(Sx, T y), d(Ax, Sx), d(By, T y), \frac{1}{2}\left(d(Sx, By) + d(Ax, T y)\right)\}$,
(d) $d(Ax, By) \leq k[d(Sx, Ty) + d(Ax, Sx) + d(By, Ty) + d(Sx, By) + d(Ax, T y)]$
for $0 \leq k < 1/3$. Then, A, B, S and T have a unique common fixed point

In **2011**, Altun, I. and Erduran, A. [7] proved fixed point theorems for monotone mappings on partial metric spaces. They proved the following result:

Let $(X,\preccurlyeq)$be partially ordered set, and suppose that there is a partial metric pon Xsuch that $(X,p)$is a completepartial metric space. Suppose $F: X \to X$is a continuous and nondecreasing mapping such that

$$p(Fx,Fy) \le \psi(max\{p(x,y),p(x,Fx),p(y,Fy),\frac{1}{2}[p(x,Fy)+p(y,Fx)]\})$$

Samet B.[26] in **2011** introduced a new class of a pair of generalized nonlinear contractions on partially ordered partial metric spaces and some coincidence and common fixed-point theorems for these contractions are proved.

Let $(X,\preccurlyeq)$ be a partially ordered set and suppose that there is a partial metricpon X such that (X,p) is a complete partial metric space. Let $F,g: X \to X$be two continuous self-mappings of X such that$FX \subseteq gX$, F is a g −non-decreasing mapping, the pair $\{F,g\}$ is partial compatible, and

$$p(Fx,Fy) \le \varphi(max\{p(gx,gy),p(gx,Fx),p(gy,Fy),\frac{1}{2}[p(gx,Fy)+p(gy,Fx)]\})$$

for all $x,y \in X$ for which $gy \preccurlyeq gx$, where a function $\varphi \in \phi$. If there exists $x_0 \in X$with$gx_0 \preccurlyeq Fx_0$, then F and g have a coincidence point, that is, there exists $x \in X$ such that$Fx = gx$. Moreover, we have

$$p(x,x) = p(Fx,Fx) = p(gx,gx) = 0.$$

Karapınar, E, Yuksel, U [20] in 2011 proved some well-known results on common fixed point are and generalized to the class of partial metric spaces.

Suppose that (X,p) is a complete PMS and T, S are self-mappings on X. If there exists an $r\in[0,1)$ such that

$$p(Tx,Sy) \le rM(x,y)$$

for any $x,y \in X$, where

$$M(x,y) = max\{p(Tx,x),p(Sy,y),p(x,y),\frac{1}{2}[p(Tx,y)+p(Sy,x)]\}$$

then there exists$z \in X$such that $Tz = Sz = z$.

II. Preliminaries

We recall the notion of a partial metric space and some of its properties which will be useful later on.

Definition 2.1.A partial metric is a function $p: X \times X \to [0,\infty)$ satisfying the following conditions:
(P1)$p(x,y) = p(y,x)$,
(P2)$p(x,x) = p(x,y) = p(y,y)$, Ifthen $x = y$,
(P3)$p(x,x) \le p(x,y)$,
(P4) $p(x,z) + p(y,y) \le p(x,y) + p(y,z)$,for all $x,y,z \in X$.
Then(X,p) is called a partial metric space.

Example 2.2[1]If$X = \{[a,b]: a,b \in R, a \le b\}$ then$p([a,b],[c,d]) = max\{b,d\} - min\{a,c\}$defines a partial metric $pon\ X$.
If p is a partial metric on X, then the function $dp: X \times X \to [0,\infty)$ given by

$$dp(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

is a metric on X. Also, each partial metric p on X generates a T_0 topology τ_p on X with a base of the family of open p-balls$\{B_p(x,\varepsilon): x \in X, \varepsilon > 0\}$ where

$$B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$$

$f$or all $x \in X$ and $\varepsilon > 0$. Similarly, closed p-ball is defined as

$$B_p(x,\varepsilon) = \{y \in X : p(x,y) \le p(x,x) + \varepsilon\}.$$

Definition 2.3.[1,7] Let (X,p) be a partial metric space.
(i)A sequence $\{x_n\}$ in X converges to $x \in X$ whenever

$$\lim_{n\to\infty} p(x,x_n) = p(x,x)$$

(ii)A sequence $\{x_n\}$ in$X$ is called Cauchy whenever $\lim_{n,m\to\infty} p(x_n,x_m)$ exists (and finite),

(iii) $(X, p)$is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$, that is,$\lim_{n,m\to\infty} p(x_n, x_m) = p(x, x)$.

(iv)A mapping $f: X \to X$ is said to be continuous at $x_0 \in X$ for each $\varepsilon > 0$ there exists $\delta > 0$ such that $f(B(x_0, \delta)) \subset B(f(x_0), \varepsilon)$.

Lemma 2.4. [1,7] Let (X, p) be a partial metric space.
(a) A sequence $\{x_n\}$is Cauchy if and only if $\{x_n\}$is a Cauchy sequence in the metric space(X, d_p),
(b) $(X, p)$is complete if and only if the metric space (X, d_p),is complete. Moreover,

$$\lim_{n\to\infty} d_p(x, x_n) = 0 \Leftrightarrow \lim_{n\to\infty} p(x, x_n) = \lim_{n,m\to\infty} p(x_n, x_m) = p(x, x).$$

In 2002, Aamri and Moutaawakil [30] introduced the (E.A)-property and obtained common fixed points for two mappings.

Definition 2.5.[30]Let (X, p) be a partial metric space. Two self-maps f andg on X are said to satisfy the (E.A)-property if there exists a sequence $\{x_n\}$ in$X$ such that $\{fx_n\}$ and $\{gx_n\}$are convergent to some $t \in X$ and $p(t, t) = 0$.

Example2.6: Let $X = [0, 4]$be a partial metric space with

$$p(x, y) = \begin{cases} |x - y| if x, y \in [0, 2] \\ max\{x, y\}, otherwise \end{cases}$$

Let $f, g: X \to X$ be defined by

$$fx = \begin{cases} 2 - x , x \in [0,1], \\ \frac{2 - x}{2}, x \in (1,2], \\ 0, \quad x \in (2, 4], \end{cases}$$

$$gx = \begin{cases} \frac{3 - x}{2}, x \in [0,1] \\ \frac{x}{2}, x \in (1,4] \end{cases}$$

For a decreasing sequence $\{x_n\}$ in X such that $x_n \to 1, gx_n \to \frac{1}{2}$, $fx_n \to \frac{1}{2}, gfx_n = \frac{4+x_n}{4} \to \frac{5}{4}$ and $, fgx_n = \frac{4-x_n}{4} \to \frac{3}{2}$ .So f and g are noncompaitible. Note that there exists a sequence$\{x_n\}$ in X such that

$$lim_{n\to\infty} f\, x_n = lim_{n\to\infty}\, gx_n = 1. \in X. \text{ Take } x_n = 1, \text{ for each} n \in N.$$

Hence f and g satisfy the(E.A)-property.

Definition 2.7. [31]Let$X$ be a non empty set and$f, g : X \to X$.If$w = fx = gx$, for some$x \in X$, then$x$ is called a coincidence point of f and g, and w is called a point of coincidence of f and g.
If $w = x$, then$x$ is a common fixed point of f and g.

Definition 2.8.[31]Let S and T be two self-maps defined on a non-emptyset$X$. Then $S$and T are said to be weakly compatible if they commute at every coincidence point i.e. if $St = Tt$for some$t \in X$, then$STt = TSt$.

Recently, Ćirić et al. [18] established a common fixed point result for two pairs of weaklycompatible mappings satisfying generalized contractions on a partial metric space. For this,denote by $\Phi$the set of non-decreasing continuous functions $\phi: R \to R$satisfying:
(a) $0 < \phi(t) < t$for all$t > 0$,
(b) the series$\sum_{n\geq 1} \emptyset^n(t)$converge for all$t > 0$.
The result [15] is the following.

Theorem 2.9. Suppose thatA, B, S, and$T$ are self-maps of a complete partial metricspace (X, p) such that
$AX \subseteq TX$, $BX \subseteq SX$ and$p(Ax, By) \leq \phi(M(x, y))$
$for all x, y \in X, where \phi \in \Phi and$

$$M(x,y) = max\left\{p(Sx,Ty), p(Ax,Sx), p(By,Ty), \frac{1}{2}[p(Sx,By) + p(Ax,Ty)]\right\}$$

If one of the ranges $AX, BX, TX$and SX is a closed subset of(X,p), then
(i) $A$and$S$ have a coincidence point,
(ii) $B$and$T$ have a coincidence point.
Moreover, if the pairs $\{A,S\}$ and $\{B,T\}$ are weakly compatible, then $A,B,T,$ and Shave a unique common fixed point.

3. Main Results

The following lemmas will be used in the proofs of the main results.

Lemma 3.1. [6, 21] Let(X,p) be a partial metric space. Then
(a) If $p(x,y) = 0$ then $x = y$,
(b) If$x \neq y$, then $p(x,y) > 0$

Lemma 3.2.[6, 21] Let(X,p) be a partial metric space and$x_n \to z$with$p(z,z) = 0$.
Then $lim_{p\to\infty} p(x_n,\gamma) = p(z,\gamma)$ for all $\gamma \in X$.

Theorem 3.3. Let A,B,S and$T$ be any self-maps of a partial metric space(X,p) satisfying the following conditions;
$(C_1) AX \subseteq TX, BX \subseteq SX,$ (1)
(C_2) Given$\epsilon > 0$, there exists a $\delta > 0$ such that for all x,yin$X$
$\varepsilon < M(x,y) < \varepsilon + \delta \Rightarrow p(Ax,By) < \varepsilon$ (2)
where

$$M(x,y) = max\left\{p(Sx,Ty), p(Ax,Sx), p(By,Ty), \frac{1}{2}\big(p(Sx,By) + p(Ax,Ty)\big)\right\}$$

(C_3) for all $x,y \in X$ with $M > 0 \Longrightarrow p(Ax,By) < M(x,y)$
(C_4)

$$p(Ax,By) < max\{\alpha_1 [p(Sx,Ty) + p(Ax,Sx) + p(By,Ty)] + \alpha_2 [p(Sx,By) + p(Ax,Ty)]\}$$

for$0 \leq \alpha_1 < \frac{1}{2}, 0 \leq \alpha_2 < \frac{1}{2}$ (3)
If one of $AX, BX, SX and TX$is a closed subset of X, then
(i) $A$and$S$have a coincidence point,
(ii)$B and T$have coincidence point.
Moreover,if $A$and Sas well as$B$and $T$are weakly compatible, then$A,B,S$and Thave a unique common fixed point.

Proof: Let x_0 be an arbitrary point in X.since$AX \subseteq TX$, there exists $x_1 \in X$ such that$Tx_1 = Ax_0$.
Since $BX \subseteq SX$, there exists $x_2 \in X$ such that $Sx_2 = Bx_1$.continuing this process, we can construct Sequences$\{x_n\}$ and $\{y_n\}$ in X defined by:

$$y_{2n} = Tx_{2n+1} = Ax_{2n}, y_{2n+1} = Sx_{2n+2} = Bx_{2n+1}, \forall n \in N \quad (4)$$

Suppose $p(y_{2n}, y_{2n+1}) = 0$ for some. Then $y_{2n} = y_{2n+1}$ implies that

$$Ax_{2n} = Tx_{2n+1} = Bx_{2n+1} = Sx_{2n+2},$$

So T and B have a coincidence point. Further, if $p(y_{2n+1}, y_{2n+2}) = 0$ for some n then

$$Ax_{2n+2} = Tx_{2n+3} = Bx_{2n+1} = Sx_{2n+2},$$

so A and S have a coincidence point.
For the rest, assume that $p(y_n, y_{n+1}) \neq 0$ for all $n \geq 0$.If for some$x,y \in X$, $M(x,y) = 0$, then we get that $Ax = Sx$ and$By = Ty$,so we proved (I) and (II).
If $M(x,y) > 0$ for all $x,y \in X$, then by (C_3)

$$p(Ax,By) < M(x,y) \text{for all } x,y \in X, \quad (5)$$

Hence we have

$$p(y_{2p}, y_{2p+1}) < M(x_{2p}, x_{2p+1})$$
$$= max\left\{p(Sx_{2p}, Tx_{2p+1}), p(Ax_{2p}, Sx_{2p}), p(Bx_{2p+1}, Tx_{2p+1}), \frac{1}{2}[p(Sx_{2p}, Bx_{2p+1}) + p(Ax_{2p}, Tx_{2p+1})]\right\}$$
$$= max\left\{p(y_{2p-1}, y_{2p}), p(y_{2p}, y_{2p-1}), p(y_{2p+1}, y_{2p}), \frac{1}{2}[p(y_{2p-1}, y_{2p+1}) + p(y_{2p}, y_{2p})]\right\}$$

$\leq max\left\{p(y_{2p-1},y_{2p}),p(y_{2p+1},y_{2p}),\frac{1}{2}[p(y_{2p-1},y_{2p})+p(y_{2p},y_{2p+1})]\right\}$
$= max\{p(y_{2p-1},y_{2p}),p(y_{2p},y_{2p+1})\}$
Since

$$p(y_{2p-1},y_{2p+1})+p(y_{2p},y_{2p})\leq p(y_{2p-1},y_{2p})+p(y_{2p},y_{2p+1})$$

It is $max\{p(y_{2p-1},y_{2p}),p(y_{2p},y_{2p+1})\}=p(y_{2p},y_{2p+1})$ is excluded. It follows that

$$p(y_{2p},y_{2p+1})<M(x_{2p},x_{2p+1})\leq p(y_{2p-1},y_{2p})\text{forall } p\geq 1 \quad (6)$$

Similarly, one can find

$$p(y_{2p+2},y_{2p+1})<M(x_{2p+2},x_{2p+1})\leq p(y_{2p+1},y_{2p})\ forall p\geq 0 \quad (7)$$

We deduce that

$$p(y_n,y_{n+1})<p(y_{n-1},y_n)\text{for all } n\geq 1.$$

Thus $\{p(y_n,y_{n+1})\}_{n=0}^{\infty}$ is a decreasing sequence which is bounded below by 0. Hence, it converges to some $L\in[0,\infty)$ i.e.

$$lim_{n\to\infty}p(y_n,y_{n+1})=L \quad (8)$$

We claim that L=0. If $l>0$, then from (8), there exists $\delta>0$ and a natural no.$m\geq 1$ such that $n\geq mL<d(y_n,y_{n+1})<L+\delta$. In particular, from this and (6)

$$L<M(x_{2m},x_{2m+1})<L+\delta.$$

Now by using (3), we get that $p(Ax_{2m},Bx_{2m+1})=p(y_{2m},y_{2m+1})\leq L$ which is a contradiction. Thus $L=0$, that is,

$$lim_{n\to\infty}p(y_n,y_{n+1})=0 \quad (9)$$

We claim that$\{y_n\}$ is a Cauchy sequence in the partial metric space (X,p). From Lemma 3.1, we need to prove that$\{y_n\}$ is Cauchy in the metric space (X,d_p). We argue by contradiction. Then there exists$\varepsilon>0$ and a subsequence$\{y_{n(i)}\}$ of$\{y_n\}$ such that$d_p(y_{n(i),}y_{n(i+1)})>4\varepsilon$, select$\delta$ in (C2) as $0<\delta\leq\varepsilon$.By definition if the metric$d_{p'}$

$$d_p(x,y)\leq 2p(x,y)\text{for all} x,y\in X,$$

So $p(y_{n(i),}y_{n(i+1)})>2\varepsilon$. Since$lim_{n\to\infty}p(y_{n'}y_{n+1})=0$, hence there exists$N\in\mathbb{N}$ such that

$$p(y_{n'}y_{n+1})<\frac{\delta}{6}\text{whenever} n\geq N.$$

Let$n(i)\geq N$. Then, there exist integers$m(i)$ satisfying$n(i)<m(i)<n(i+1)$ such that

$$p(y_{n(i)'}y_{m(i)})\geq\varepsilon+\frac{\delta}{3}$$

If not, then by triangle inequality (which holds even for partial metrics)

$$p(y_{n(i)'}y_{n(i+1)})\leq p(y_{n(i)'}y_{n(i+1)-1})+p(y_{n(i+1)-1'}y_{n(i+1)})$$

$<\varepsilon+\frac{\delta}{3}+\frac{\delta}{6}<2\varepsilon,$
It is a contradiction. Without loss of generality, we can assume$n(i)$ to be odd. Let$m(i)$ be the smallest even integer such that

$$p(y_{n(i)'}y_{m(i)})\geq\varepsilon+\frac{\delta}{3} \quad (10)$$

Then

$$p(y_{n(i)'}y_{m(i)-2})\geq\varepsilon+\frac{\delta}{3}$$

and

$$\varepsilon+\frac{\delta}{3}\leq p(y_{n(i)'}y_{m(i)})\leq p(y_{n(i)'}y_{m(i)-2})+p(y_{m(i)-2'}y_{m(i)-1})+p(y_{m(i)-1'}y_{m(i)})$$

$<\varepsilon+\frac{\delta}{3}+\frac{\delta}{6}+\frac{\delta}{6}=\varepsilon+2\frac{\delta}{3}.$ (11)
Also,$p(y_{n(i)'}y_{m(i)})\leq M(x_{n(i)+1'}x_{m(i)+1})<\varepsilon+2\frac{\delta}{3}+\frac{\delta}{6}<\varepsilon+\delta$, that is,

$$\varepsilon<\varepsilon+\frac{\delta}{3}\leq M(x_{n(i)+1'}x_{m(i)+1})<\varepsilon+\delta.$$

In view of (C2), this yields that$p(y_{n(i),+1}y_{m(i)+1})\leq\varepsilon$. But then

$$p(y_{n(i)'}y_{m(i)})\leq p(y_{n(i)'}y_{n(i)+1})+p(y_{n(i)+1'}y_{m(i)+1})+p(y_{m(i)+1}y_{m(i)})$$

$<\frac{\delta}{6}+\varepsilon+\frac{\delta}{6}=\varepsilon+\frac{\delta}{3}$

which contradicts (10). Hence $\{y_n\}$ is a Cauchy sequence in the metric space (X, d_p), so also in the partial metric space (X, p) which is complete. Thus there exists a point y in X such that from lemma 3.1, 3.2, and (9)

$$p(y,y) = lim_{n\to\infty} p(y_n, y) = lim_{n\to\infty} p(y_n, y_n) = 0 \quad (12)$$

This implies that

$$lim_{n\to\infty} p(y_{2n}, y) = lim_{n\to\infty} p(y_{2n-1}, y) = 0 \quad (13)$$

Thus from (13) we have

$$lim_{n\to\infty} p(Ax_{2n}, y) = lim_{n\to\infty} p(Tx_{2n+1}, y) = 0 \quad (14)$$

And

$$lim_{n\to\infty} p(Bx_{2n-1}, y) = lim_{n\to\infty} p(Sx_{2n}, y) = 0 \quad (15)$$

Now we can suppose, without loss of generality, that SX is a closed subset of the partial metric space(X, p). From (15), there exists $u \in X$ such that $y = Su$.we claim that$p(Au, y) = 0$.

Suppose $p(Au, y) > 0$.By (P4) and (C4), we get

$$\begin{aligned} p(y, Au) &\le p(y, Bx_{2n+1}) + p(Au, Bx_{2n+1}) - p(Bx_{2n+1}, Bx_{2n+1}) \\ &\le p(y, Bx_{2n+1}) + p(Au, Bx_{2n+1}) \\ &\le p(y, Bx_{2n+1}) + max\{\alpha_1[p(y, y_{2n}) + p(Au, y) + p(y_{2n+1,} y_{2n})] \\ &\quad +\alpha_2[p(y, y_{2n+1}) + p(Au, y_{2n})]\} \\ &\le p(y, Bx_{2n+1}) + max\{\alpha_1[p(y, y_{2n}) + p(Au, y) + p(y_{2n+1,} y_{2n})] \\ &\quad + \alpha_2[p(y, y_{2n+1}) + p(Au, y) + p(y, y_{2n}) - p(y, y)]\} \end{aligned}$$

Letting$n \to \infty$ in the above inequality and using (12)-(15), we obtain

$$0 < p(y, Au) \le max\{\alpha_1 p(Au, y) + \alpha_2 p(Au, y)\} < p(Au, y)$$

it is a contradiction since$0 \le \alpha_1 < \frac{1}{2}, 0 \le \alpha_2 < \frac{1}{2}$. Thus, by Lemma 2.1, we deduce that

$$p(Au, y) = 0 \ and y = Au. \quad (16)$$

Since$y = Su$, then$Au = Su$, that is, u is a coincidence point of$A and S$, so we proved (I).
From$AX \subseteq TX$ and (16), we have$y \in TX$. Hence we deduce that there exists$v \in X$ such that$y = Tv$. We claim that$p(Bv, y) > 0$. suppose, to the contrary, that$p(Bv, y) > 0$. From (C4) and (16), we have.

$$\begin{aligned} 0 < p(y, Bv) = p(Au, Bv) &\le max\{\alpha_1[p(Su, Tv) + p(Au, Su) + p(Bv, Tv)] \\ &\quad + \alpha_2[p(Su, Bv) + p(Au, Tv)]\} \\ &= max\{\alpha_1[p(y, y) + p(y, y) + p(Bv, y)] \\ &\quad + \alpha_2[p(y, Bv) + p(y, y)]\} \\ &= max\{\alpha_1 p(Bv, y) + \alpha_2 p(Bv, y)\} \end{aligned}$$

as$y = Su = Au = Tv and p(y, y) = 0$. Since$0 \le \alpha_1 < \frac{1}{2}, 0 \le \alpha_2 < \frac{1}{2}$, this implies that

$$p(Bv, y) < p(Bv, y),$$

which is a contradiction. Then, we deduce that

$$p(Bv, y) = 0 \ and y = Bv = Tv. \quad (17)$$

that is,v is a coincidence point of B and T, then (II) holds.
Since the pair $\{A, S\}$ is weakly compatible, from (16), we have$Ay = ASu = SAu = Sy$.
We claim that$p(Ay, y) = 0$. Suppose, to the contrary, that$p(Ay, y) > 0$. We have

$$\begin{aligned} p(Ay, y) &\le p(Ay, y_{2n+1}) + p(y_{2n+1}, y) \\ &= p(Ay, Bx_{2n+1}) + p(y_{2n+1}, y) \\ &\le p(y_{2n+1}, y) + max\{a[p(Sy, Tx_{2n+1}) + p(Ay, Sy) + p(Bx_{2n+1}, Tx_{2n+1)}], \\ &\quad b[p(Sy, Bx_{2n+1}) + (Ay, Tx_{2n+1})]\} \\ &= p(y_{2n+1}, y) + max\{a[p(Ay, y_{2n}) + p(Ay, Ay) + p(y_{2n+1} + y_{2n)}], \\ &\quad b[p(Ay, y_{2n+1}) + p(Ay, y_{2n})]\}. \end{aligned}$$

Using (12) and (p2), we get letting $n \to +\infty$

$$0 < p(Ay,y) \leq max\{2ap(Ay,y), 2bp(Ay,y)\} < p(Ay,y)$$

a contradiction. Then we deduce that

$p(Ay,y) = 0 \; and Ay = Sy = y.$ (18)

Since the pair $\{B,T\}$ is weakly compatible, from (17), we have $By = BTv = TBv = Ty$. We claim that $p(By,y) = 0$. Suppose, to the contrary, that $p(By,y) > 0$, then by (C4) and (3.3.18), we have

$$0 < p(y,By) = p(Ay,By) \leq max\{a[p(Sy,Ty) + p(Ay,Sy) + p(By + Ty)], b[p(Sy,By) + p(Ay,Ty)]\}$$
$$= max\{a[p(y,By) + p(y,y) + p(By,By)], b[]p(y.By) + p(y,By)]\}$$

$\leq max\{2a, 2b\}p(By,y),$

since $p(y,y) = 0$. Thus, we get

$p(y,by) = 0 \; and By = Ty = Y.$ (19)

Now, combining (18) and (19), we obtain.

$$y = Ay = By = Sy = Ty,$$

that is, y is a common fixed point of $A,B,S, and Y with p(y,y) = 0$.

Now we prove that uniqueness of a common fixed point. Let us suppose that $z \in X$ is a common fixed point of $A,B,S, and T$ such that $p(z,y) > 0$. Using (iv), we get

$$p(y,z) = p(Ay,Bz)$$
$$\leq max\{a[p(Ay,Bz) + p(Ay,Ay) + p(Bz,Bz)], b[p(Ay,Bz) + p(Az,By)]\}$$

$= max\{a[p(y,z) + p(y,y) + p(z,z), 2bp(y,z)\}$

$\leq max\{2a, 2b\}p(y,z)) < p(y,z),$

which is a contradiction. Then we deduce that $z = y$. Thus the uniqueness of the common fixed point is proved. The proof is completed.

Corollary3.4. Let A,B,S and T be any self-maps of a partial metric space (X,p) satisfying the following conditions;

$(C_1) AX \subseteq TX, BX \subseteq SX,$ (20)

(C_2) Given $\epsilon > 0$, there exists a $\delta > 0$ such that for all x, y in X

$\varepsilon < M(x,y) < \varepsilon + \delta \Rightarrow p(Ax,By) < \varepsilon$ (21)

where $M(x,y) = max\left\{p(Sx,Ty), p(Ax,Sx), p(By,Ty), \frac{1}{2}\left(p(Sx,By) + p(Ax,Ty)\right)\right\}$

(C_3) for all $x,y \in X$ with $M > 0 \Longrightarrow p(Ax,By) < M(x,y)$

$(C_4) p(Ax,By) < k[p(Sx,Ty) + p(Ax,Sx) + p(By,Ty) + \; p(Sx,By) + p(Ax,Ty)]$

for $0 \leq \alpha_1 < \frac{1}{2}, 0 \leq \alpha_2 < \frac{1}{2}$ (22)

If one of $AX, BX, SX and TX$ is a complete subspace of X, then

(i) A and S have a coincidence point

(ii) $B and T$ have coincidence point

Moreover, if A and S, as well as, B and T are weakly compatible, then A,B,S and T have a unique common fixed point.

Theorem 3.5. Let A,B,S and T be any self-maps of a partial metric space (X,p) satisfying the following conditions;

(i) $AX \subseteq TX, BX \subseteq SX,$ (23)

(ii)

$$p(Ax,By) < max\left\{\alpha_1[p(Sx,Ty) + p(Ax,Sx) + p(By,Ty)] + \alpha_2\left[\frac{p(Sx,By) + p(Ax,Ty)}{2}\right]\right\}$$

For $0 \leq \alpha_1 < 1, \; 1 \leq \alpha_2 < 2.$ (24)

Let one of the mappings $(A,S) or (B,T)$ be weakly compatible, satisfying property (E. A.). If the range of one of the mappings be a complete subspace of X, then A,B,S and T have a unique common foixed point.

Proof: Let B and T satisfy property E.A. Then $\exists$ a sequence $\{x_n\}$ in X such that $Bx_n \to t$ and $Tx_n \to t$ for some t in X. Since $BX \subseteq SX$, for each $x_n, \exists y_n$ in X, such that $Bx_n = Sy_n$. Thus $Bx_n \to t, Tx_n \to t$ and $Sy_n \to t$. We claim that $Ay_n \to t$.if not, there exists a subsequence $\{Ay_n\}$, a positive integer M and a number r>0 such that for each $m \geq M$, we have

$p(Ay_m, t) \geq r, p(Ay_m, Bx_m) \geq r,$ (25)

$$p(Ay_m, Bx_m) < max\left\{\alpha_1[p(Sy_m, Tx_m) + p(Ay_m, Sy_m) + p(Bx_m, Tx_m)] + \alpha_2\left[\frac{p(Ay_m, Tx_m) + p(Bx_m, Sy_m)}{2}\right]\right\} \quad (26)$$

$< p(Ay_m, Sy_m)$ (27)

a contradiction. Hence $Ay_n \to t$. Now suppose that SX is a complete subspace of X.Then, since$Sy_n \to t$, then ∃apoint u in X, such that $t = Su$.

If $Au \neq Su$, the inequality,

$$p(Au, Bx_n) < max\left\{\begin{matrix}\alpha_1[p(Su, Tx_n) + p(Au, Su) + p(Bx_n, Tx_n)] + \\ \alpha_2\frac{[p(Au, Tx_n) + p(Bx_n, Su)]}{2}\end{matrix}\right\} \quad (28)$$

On taking $n \to \infty$, yields

$$p(Au.Su) < p(Au, Su),$$

a contradiction. Hence $Au = Su$. Since $AandS$ are weakly compatible so it implies that

$$ASu = SAu$$

and so$Au = ASu = SAu = Su$.

On the other hand, since $AX \subset TX$, there exists a point $w \in X$, such that $Au = Tw$.

We assert that $Tw = Bw$.

If $Bw \neq Tw$, then by (24) we get

$$p(Au, Bw) < max\left\{\alpha_1[p(Su, Tw) + p(Au, Su) + p(Bw, Tw)] + \alpha_2\left[\frac{p(Au, Tw) + p(Bw, Su)}{2}\right]\right\}$$

$$< p(Bw, Au) \quad (29)$$

acontradiction hence $Au = Bw = Tw = Su$, which shows that the pair (A, S) and (B, T) have a pair of coincidence u and w respectively.. The proof is similar if we consider the case when pair (A, S) enjoys property$(E.A.)$.

Now by weak compatibility property of B and T, it implies that $BTw = TBw$ and $BBw = BTw = TBw = TTw$.suppose that $Au \neq AAu$.so we have from (24),

$$p(Au, AAu) = p(AAu, Bw) < max\left\{\begin{matrix}\alpha_1[p(SAu, Tw) + p(AAu, SAu) + p(Bw, Tw)] \\ +\alpha_2\frac{[p(AAu, Tw) + p(Bw, SAu)]}{2}\end{matrix}\right\}$$

$< p(AAu, Au)$. (30)

which is a contradiction. Thus $Au = AAu = SAu$ and Au is a common fixed of B and T.

The proof is similar when TX is assumed to be complete subspace of X.The cases in which AXor BX is a complete subspace of X are similar to the cases in which TX or SX respectively be complete since $AX \subset TXandBX \subset SX$. The uniqueness of the common fixed point follows easily from (24). Hence the theorem.

Theorem3.6. Let A, B, S and$T$ be any weakly compatible self-maps of a partial metric space(X, p) satisfying the following conditions;

(i) $AX \subset TX, BX \subset SX,$ (31)

(ii)

$$p(Ax, By) < max\left\{\alpha_1[p(Sx, Ty) + p(Ax, Sx) + p(By, Ty)] + \alpha_2\left[\frac{p(Sx, By) + p(Ax, Ty)}{2}\right]\right\}$$

For $0 \leq \alpha_1 < 1,\ 1 \leq \alpha_2 < 2.$ (32)

Let one of the mappings $(A, S)or(B, T)$ be non-compatible, satisfying property (E. A.). If the range of one of the mappings be a complete subspace of X, then A, B, S and T have a unique common foixed point and the fixed point is a point of discontinuity.

Proof: Let B and T be noncompaitible maps,so there exists a sequence in X such that

$lim_{n\to\infty} Bx_n = tandlim_{n\to\infty} Tx_n = t$

(33)

For some $t \in X$, but $lim_{n\to\infty}p(BTx_n, TBx_n)$ is either nonzero or nonexistent$\{x_n\}$. Since$BX \subset SX$, for each x_n, there exists a $y_n \in X$ such that $Bx_n = Sy_n$.Thus

$$Bx_n \to t, Tx_n \to t \text{ and } Sy_n \to t.$$

We claim that $Ay_n \to t$.If not, there exists a subsequence$\{Ay_m\}$of$\{Ay_n\}$, a positive inmteger M and a number $r > 0$ such that for each $m \geq M$,we have

$p(Ay_m, t) \geq r$, $p(Ay_m, Bx_m) \geq r$(34)

$$p(Ay_m, Bx_m) < max\left\{\alpha_1[p(Sy_m, Tx_m) + p(Ay_m, Sy_m) + p(Bx_m, Tx_m)] + \alpha_2\left[\frac{p(Sy_m, By) + p(Ay_m, Ty)}{2}\right]\right\}$$
(35)

$$< p(Ay_m, Sy_m) \quad (36)$$

a contradiction. Hence $Ay_m \to t$. Suppose that SX is a complete subspace of X. Then since $Sy_n \to t$ there exists a point u in $Xsuchthat$t = Su.

If $Au \neq Su$, the inequality,

$$p(Au, Bx_n) < max\left\{\alpha_1[p(Su, Tx_n) + p(Au, Su) + p(Bx_n, Tx_n)] + \alpha_2\left[\frac{p(Su, Bx_n) + p(Au, Tx_n)}{2}\right]\right\} \quad (37)$$

On taking$n \to \infty$, yields$p(Au, Su) < p(Au, Su)$ a contradiction. Hence $Au = Su$. Since A and S are weakly compatible so it implies that $ASu = SAu$ and then $AAu = ASu = SAu = SSu$.

On the other hand, since $AX \subset TX$, there exists a point $w \in X$, such that $Au = Tw$.We assert that $Tw = Bw$. If $Bw \neq Tw$, then by (32), we get

$$p(Au, Bw) < max\left\{\alpha_1[p(Su, Tw) + p(Au, Su) + p(Bw, Tw)] + \alpha_2\left[\frac{p(Au, Tw) + p(Bw, Su)}{2}\right]\right\}$$
$$< p(Bw, Au) \quad (38)$$

a contradiction. Hence $Au = Su = Bw = Tw$, which shows pair $(A, S)$and(B, T) have a point of coincidence respectively. The proof is similar if we consider the case when pair (A, S) enjoys property (E.A.)

Now by weak compatibility of B and T, it implies that $BTw = TBw$ and $BBw = BTw = TBw = TTw$. Now, suppose that $Au \neq AAu$.Sowe have from (32)

$$p(Au, AAu) = p(AAu, Bw)$$
$$< max\left\{\alpha_1[p(SAu, Tw) + p(AAu, SAu) + p(Bw, Tw)] + \alpha_2\left[\frac{p(AAu, Tw) + p(Bw, SAu)}{2}\right]\right\} \quad (39)$$
$$< p(AAu, Au) \quad (40)$$

which is a contradiction. Thus $Au = AAu = SAu$, then Au is a common fixed point of $A$and S.Similarly$Au = Bw$ is a common fixed point of B and T. The proof is similar when TX is assumed to be complete subspaceof$X$. The cases in which AX or BX is complete subspace of X are similar to the cases in which TX or SX respectively be complete since $AX \subset TX$and $BX \subset SX$ .Uniqueness of the common fixed point follows easily.

We have to show now that the mappings are discontinuous at the common fixed point. Let us suppose that B is continuous at common fixed point t, such that $t = Au = Bw$.So on taking the sequence $\{x_n\}$as taken in (32), we have

$$\lim_{n\to\infty} BTx_n = Bt = t.$$

By weak compatibility property of $B$and T, it follows that $BTx_n = TBx_n$.

On letting $n \to \infty$, this gives us

$$\lim_{n\to\infty} BTx_n = \lim_{n\to\infty} TBx_n = Bt = t.$$

Thus $p(BTx_n, TBx_n) = p(Bt, Bt)$ =0,

which contradicts the fact that $lim_{n\to\infty}\, p(BTx_n, TBx_n)$ is either nonzero or nonconsistent for the sequence$\{x_n\}$ of (32). Hence B is discontinuous at the fixed point.

Now, suppose that T is continuous, then for the sequence $\{x_n\}$of (32), we get

$$lim_{n\to\infty}\, TBx_n = Tt = t \text{and } \lim_{n\to\infty} TTx_n = Tt = t.$$

Hence the inequality, in view of these limits, gives us;

$$p(At, BTx_n) < \max\{\alpha_1[p(St, TTx_n) + p(At, St) + p(BTx_n, TTx_n)] + \alpha_2[p(At, TTx_n) + p(BTx_n, St)]/2\} \quad (41)$$

whichis a contradiction, unless

$$\lim_{n\to\infty} BTx_n = TTx_n = Tt = t.$$

But $\lim_{n\to\infty} BTx_n == Tt = t \; and \; \lim_{n\to\infty} TBx_n = Tt = t$ which contradicts the fact that $p(BTx_n, TBx_n)$is either nonzero or nonconsistent. Hence both B and T are discontinuous at the common fixed point.Similarly, it can be shown that A and S are also discontinuous at the common fixed point. Thus all the self-mapsA, B, S and T are discontinuous at the common fixed point. Hence the theorem is established.

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