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# Discrete Delay Heat Equation By Partial Alpha-Beta Difference Operators

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***ABSTRACT:*** Partial difference equation which extends its application in heat equation is taken for study by the application of $\alpha-\beta$ difference operator in this paper. With Fourier law of cooling as its basis, the heat propagation for long rod, thin plate and medium up to three variables with validation done using MATLAB is carried out.

***KEYWORDS:*** Partial difference equation, partial difference operator and discrete heat equation.
***AMS Subject classification:*** 39A70, 39A10, 47B39, 80A20.

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## I. Introduction

The difference operator $\underset{\alpha}{\Delta}$ was introduced in 1984, by Jerzy Popenda [5] which is defined on $u(k)$ as $\underset{\alpha}{\Delta}u(k) = u(k+1) - \alpha u(k)$. The same definition was continued to $\underset{\alpha(\ell)}{\Delta}$ in 2011, by M.Maria Susai Manuel, et.al, [6], defined as $\underset{\alpha(\ell)}{\Delta} v(k) = v(k+\ell) - \alpha v(k)$. The author G. Britto Antony Xavier, et.al,[4] introduced the k-difference operator $\underset{k(l)}{\Delta}$ with variable coefficients as $\underset{k(\ell)}{\Delta} v(k) = v(k+\ell) - kv(k)$ in 2016. Similar types of difference operators and applications are developed in [1],[3],[7],[8],[9]. Here, we extend the theory of alpha difference operator to partial alpha-beta difference equation and apply it in heat equation.

Let $\beta \neq 0$, $l = (\ell_1, \ell_2, \ell_3, ..., \ell_n) \neq 0$ and $v(k)$ be a real valued function of n variables $k = (k_1, k_2, ..., k_n)$. Then, we have $\beta$-difference operator

$$\underset{\beta(\ell)}{\Delta} v(k) = v(k_1+\ell_1, k_2+\ell_2, ..., k_n+\ell_n) - \beta v(k_1, k_2, ..., k_n). \qquad (1)$$

This operator $\underset{\beta(\ell)}{\Delta}$ becomes partial $\beta$-difference operator if some $\ell_i = 0$ but not all $\ell_j$. A first order linear partial $\beta$-difference equation is

$$\underset{\beta(\ell)}{\Delta} v(k) = u(k). \qquad (2)$$

The equation $\underset{\beta(\ell)}{\overset{-1}{\Delta}} u(k) = v(k)$ satisfies $(2)$ as $v(k)$ is a solution which is numerically intended by

$$v(k) - \beta^m v(k-m\ell) = \sum_{r=1}^{m} \beta^{r-1} u(k-r\ell) = \underset{\beta(\ell)}{\overset{-1}{\Delta}} u(k)\Big|_{k-m\ell}^{k}, \qquad (3)$$

where $k - r\ell = (k_1 - r\ell_1, k_2 - r\ell_2, ..., k_m - r\ell_m)$, $m > 0$. Equation (3) is the basic finite inverse principle with respect to $\underset{\beta(\ell)}{\Delta}$ [2]. Here, we apply the alpha-beta partial difference operator in heat equation with delay.

## II. Discrete Heat Equation Of A Long Rod

Let us take a very long rod with $v(k_1, k_2)$ as temperature at position $k_1$ and at time $k_2$. Even though we have assumed that heat flow is instantaneous, in reality, it takes time for heat to flow from one point $k_1$ to its neighbouring points $k_1-\ell_1$ and $k_1+\ell_1$ in one dimensional flow. Let $\gamma$ be the positive diffusion rate constant of rod. By denoting $\underset{\alpha(\pm\ell_1,0)}{\Delta} = \underset{\alpha(\ell_1,0)}{\Delta} + \underset{\alpha(-\ell_1,0)}{\Delta}$, taking $n=2$ in $(1)$ and Cooling law of Fourier, the discrete delay heat equation of rod is

$$\underset{\beta(0,\ell_2)}{\Delta} v(k_1,k_2) = \gamma \underset{\alpha(\pm\ell_1,0)}{\Delta} v(k_1,k_2-\sigma). \quad (4)$$

where $\sigma$ is a delay factor. Our main aim of this paper is to study and discuss the solution of the partial delay heat equation $(4)$.

**Theorem 2.**1 Let us take an integer $m>0$, and a number $\ell_2>0$ which is real such that $v(k_1,k_2-m\ell_2)$ and $\underset{\alpha(\pm\ell_1)}{\Delta} v(k_1,k_2-\sigma) = \underset{\alpha(\pm\ell_1)}{u}(k_1,k_2-\sigma)$ are well-known. Then the delay heat equation $(4)$ has a result as

$$v(k_1,k_2) = \beta^m v(k_1,k_2-m\ell_2) + \gamma\sum_{r=1}^{m}\beta^{r-1} \underset{\alpha(\pm\ell_1)}{u}(k_1,k_2-\sigma-r\ell_2). \qquad (5)$$

Proof. By representing $\underset{\alpha(\pm\ell_1,0)}{\Delta} v(k_1,k_2) = \underset{\alpha(\pm\ell_1)}{u}(k_1,k_2-\sigma)$, from $(3)$ and $(4)$, we arrive

$$v(k_1,k_2) - \beta^m v(k_1,k_2-m\ell_2) = \gamma \underset{\beta(0,\ell_2)}{\overset{-1}{\Delta}} \underset{\alpha(\pm\ell_1)}{u}(k_1,k_2-\sigma)\Big|_{k-m\ell}^{k}, \quad (6)$$

which yields $(5)$.

***Theorem 2.2*** *Denoting* $v(k_1\pm\ell_1,*) = v(k_1+\ell_1,*) + v(k_1-\ell_1,*)$ *and* $v(*,k_2\pm\ell_2) = v(*,k_2+\ell_2) + v(*,k_2-\ell_2)$. *Then, the given equations are identical:*

*(a).* $$v(k_1,k_2) = \frac{1}{\beta^m} v(k_1,k_2+m\ell_2) - \sum_{i=1}^{m}\frac{\gamma}{\beta^i}[v(k_1\pm\ell_1,k_2-\sigma+(i-1)\ell_2) - 2\alpha v(k_1,k_2+(i-1)\ell_2-\sigma)], \qquad (7)$$

*(b).* $$v(k_1,k_2) = \beta^m v(k_1,k_2-m\ell_2) + \sum_{i=1}^{m}\beta^{i-1}\gamma[v(k_1+\ell_1,k_2-i\ell_2-\sigma) + v(k_1-\ell_1,k_2-i\ell_2-\sigma) - 2\alpha v(k_1,k_2-i\ell_2-\sigma)] \qquad (8)$$

*(c).* $$v(k_1,k_2) = \frac{1}{\gamma^m} v(k_1-m\ell_1,k_2+m\ell_2+m\sigma) - \sum_{i=1}^{m}\frac{\beta}{\gamma^i} v(k_1-i\ell_1,k_2+(i-1)\ell_2+i\sigma) - \sum_{i=1}^{m}\frac{1}{\gamma^{i-1}} v(k_1-(i+1)\ell_1,k_2+(i-1)\ell_2+(i-1)\sigma) + \sum_{i=1}^{m}\frac{2\alpha}{\gamma^{i-1}} v(k_1-i\ell_1,k_2+(i-1)\ell_2)+(i-1)\sigma)) \quad (9)$$

*(d).* $$v(k_1,k_2) = \frac{1}{\gamma^m} v(k_1+m\ell_1,k_2+m\ell_2+m\sigma) - \sum_{i=1}^{m}\frac{\beta}{\gamma^i} v(k_1+i\ell_1,k_2+(i-1)\ell_2+i\sigma) + \sum_{i=1}^{m}\frac{2\alpha}{\gamma^{i-1}} v(k_1+i\ell_1,k_2+(i-1)\ell_2+(i-1)\sigma) - \sum_{i=1}^{m}\frac{1}{\gamma^{i-1}} v(k_1+(i+1)\ell_1,k_2+(i-1)\ell_2+(i-1)\sigma). \qquad (10)$$

*Proof. (a). From (4),*

$$v(k_1,k_2)=\frac{1}{\beta}v(k_1,k_2+\ell_2)-\frac{\gamma}{\beta}[v(k_1+\ell_1,(k_2-\sigma))$$
$$+v(k_1-\ell_1,k_2-\sigma)-2\alpha v(k_1,k_2+\ell_2-\sigma)]. \qquad (11)$$

*By replacing* $k_2$ *by* $k_2+\ell_2, k_2+2\ell_2,..., k_2+m\ell_2$ *in (11), we obtain expressions for* $v(k_1,k_2+i\ell_2)$ *and* $v(k_1\pm\ell_1,k_2+i\ell_2)$. *Now proof of (a) follows from (11).*

*(b). The delay heat equation (4) generates*

$$v(k_1,k_2)=\beta v(k_1,k_2-\ell_2)+\gamma[v(k_1+\ell_1,k_2-\ell_2-\sigma)$$
$$+v(k_1-\ell_1,k_2-\ell_2-\sigma)-2\alpha v(k_1,k_2-\ell_2-\sigma)]. \qquad (12)$$

*Proof follows by replacing* $k_2$ *by* $k_2-\ell_2, k_2-2\ell_2,..., k_2-m\ell_2$ *repeatedly and substituting corresponding v-values in (12).*

*(c ). A simple calculation on (4) gives the expression*

$$v(k_1,k_2)=\frac{1}{\gamma}v(k_1-\ell_1,k_2+\ell_2+\sigma)-\frac{\beta}{\gamma}v(k_1-\ell_1,k_2+\sigma)-v(k_1-2\ell_1,k_2)+2\alpha v(k_1-\ell_1,k_2).$$

*By replacing* $k_1$ *by* $k_1-\ell_1, k_1-2\ell_1,..., k_1-m\ell_1$ *and* $k_2$ *by* $k_2+\ell_2+\sigma, k_2+2\ell_2+\sigma,..., k_2+m\ell_2+\sigma$ *repeatedly, then the proof arrives.*

*(d). From* $v(k_1,k_2)=\frac{1}{\gamma}v(k_1+\ell_1,k_2+\ell_2+\sigma)-\frac{\beta}{\gamma}v(k_1+\ell_1,k_2+\sigma)$

$$-v(k_1+2\ell_1,k_2)+2\alpha v(k_1+\ell_1,k_2),$$

*Replace* $k_1$ *by* $k_1+\ell_1, k_1+2\ell_1,..., k_1+m\ell_1$ *and* $k_2$ *by* $k_2+\ell_2+\sigma, k_2+2\ell_2+\sigma,..., k_2+m\ell_2+\sigma$, *we reach the proof.*

***Example 2.3*** *The dissemination rate of rod is identified by the given example if the solution* $v(k_1,k_2)$ *of* $(4)$ *is known. If* $v(k_1,k_2)=e^{k_1+k_2}$ *is a closed form solution of* $(4)$, *then*

$\underset{\beta(0,\ell_2)}{\Delta} e^{k_1+k_2}=\gamma[\underset{\alpha(\ell_1,0)}{\Delta} e^{k_1+k_2-\sigma}+\underset{\alpha(-\ell_1,0)}{\Delta} e^{k_1+k_2-\sigma}]$, *which yields*

$e^{k_1+k_2+\ell_2}-\beta e^{k_1+k_2}=\gamma[e^{k_1+k_2+\ell_1-\sigma}+e^{k_1+k_2-\ell_1-\sigma}-2\alpha e^{k_1+k_2-\sigma}]$ *and*

$$\gamma=\frac{e^{\ell_2}-\beta}{e^{\ell_1-\sigma}+e^{-\ell_1-\sigma}-2\alpha e^{-\sigma}}. \qquad (13)$$

*We use MATLAB coding to verify (a) of Theorem* $(2.2)$ *for* $m=15$, $k_1=1$, $\ell_1=1$, $k_2=2$, $\ell_2=2$, $\alpha=2$, $\beta=3$, $\sigma=1$, $v(k_1,k_2)=e^{(k_1+k_2)}$, *and* $\gamma$ *as in* $(13)$.

$$exp(3)=((1./(3).\wedge 1).*(exp(5)))-symsum(((-13.0555764\ \ 7)./3.\wedge i).*(((exp(3+(i-1).*2)))$$
$$+exp(1+(i-1).*2)-(4.*(exp(2+(i-1).*2)))),\ i,1,1).$$

*The MATLAB codings for (b), (c)* & *(d) of Theorem* $(2.2)$ *are similar.*

## III. Heat Flow For Thin Sheet When $\gamma$ Is Constant

Secondly, let us assume $v(k_1,k_2,k_3)$ be the temperature of thin plate at real position $(k_1,k_2)$ and at time $k_3$. As in the case of rod, the partial $\beta-\alpha$ delay heat equation for the thin plate can be formulated as

$$\underset{\beta(0,0,\ell_3)}{\Delta} v(k_1,k_2,k_3)=\gamma \underset{\alpha(\pm\ell_{(1,2)})}{\Delta} v(k_1,k_2,k_3-\sigma). \qquad (14)$$

where $\underset{\alpha(\pm\ell_{(1,2)})}{\Delta}=\underset{\alpha(\ell_1,0,0)}{\Delta}+\underset{\alpha(-\ell_1,0,0)}{\Delta}+\underset{\alpha(0,\ell_2,0)}{\Delta}+\underset{\alpha(0,-\ell_2,0)}{\Delta}$ and $\sigma$ is a delay factor.

**Theorem 3.1** Let us take an integer $m>0$, and a number $\ell_3>0$ which is real such that $v(k_1,k_2,k_3-ml_3)$

and partial differences $\underset{\alpha(\pm\ell_{(1,2)})}{\Delta} v(k_1,k_2,k_3) = \underset{\alpha(\pm\ell_{(1,2)})}{u}(k_1,k_2,k_3-\sigma)$ are known functions. Then (14) has a result in numerical form as

$$v(k_1,k_2,k_3) = \beta^m v(k_1,k_2,k_3-m\ell_3) + \gamma\sum_{r=1}^{m}\beta^{r-1}\underset{\alpha(\pm\ell_{(1,2)})}{u}(k_1,k_2,k_3-\sigma-r\ell_3). \quad (15)$$

Proof. The proof of $(15)$ is identical to Theorem (5). Consider the following notations which will be used in the subsequent theorems:

$v(k_{(1,2)} \pm l_{(1,2)},*) = v(k_1 \pm \ell_1, k_2,*) + v(k_1, k_2 \pm \ell_2,*)$,

$v(k_{(2,3)} \pm l_{(2,3)},*) = v(*,k_2 \pm \ell_2, k_3) + v(*,k_2,k_3 \pm \ell_3)$,

$v(k_1 \pm \ell_1,*,*) = v(k_1 + \ell_1,*,*) + v(k_1 - \ell_1,*,*)$,

$v(*,*,k_3 \pm \ell_3) = v(*,*,k_3 - \sigma + \ell_3) + v(*,*,k_3 - \sigma - \ell_3)$.

**Theorem 3.2** Let us assume $v(k_1,k_2,k_3)$ is a solution of equation $(14)$, and $v(k_1 \pm r\ell_1, k_2 \pm r\ell_2)$ exist. Then we obtain four types of solutions to equation $(14)$.

(a). $$v(k_1,k_2,k_3) = \frac{1}{\beta^m} v(k_1,k_2,k_3+m\ell_3) - \sum_{i=1}^{m}\frac{\gamma}{\beta^i}[v(k_1 \pm \ell_1, k_2, k_3+(i-1)\ell_3-\sigma)$$
$$+ v(k_1,k_2 \pm \ell_2, k_3+(i-1)\ell_3-\sigma)] \; - 4\alpha v(k_1,k_2,k_3+(i-1)\ell_3-\sigma)] \quad (16)$$

(b). $$v(k_1,k_2,k_3) = \beta^m v(k_1,k_2,k_3-m\ell_3) + \sum_{i=1}^{m}\beta^{(i-1)}\gamma[v(k_1 \pm \ell_1, k_2, k_3-\sigma-i\ell_3)$$
$$+ v(k_1,k_2 \pm \ell_2, k_3-\sigma-i\ell_3) - 4\alpha v(k_1,k_2,k_3-\sigma-i\ell_3)] \quad (17)$$

(c). $$v(k) = \frac{1}{\gamma^m} v(k_1-i\ell_1,k_2,k_3+i\ell_3+i\sigma) - \sum_{i=1}^{m}\frac{\beta}{\gamma^i} v(k_1-i\ell_1,k_2,k_3+(i-1)\ell_3+i\sigma)$$
$$-\sum_{i=1}^{m}\frac{1}{\gamma^{i-1}} v(k_1-(i+1)\ell_1,k_2,k_3+(i-1)\ell_3+(i-1)\sigma)$$
$$-\sum_{i=1}^{m}\frac{1}{\gamma^{i-1}} v(k_1-i\ell_1,k_2 \pm \ell_2,k_3+(i-1)\ell_3+(i-1)\sigma)$$
$$-\sum_{i=0}^{m}\frac{4\alpha}{\gamma^{i-1}} v(k_1-i\ell_1,k_2,k_3+(i-1)\ell_3+(i-1)\sigma) \quad (18)$$

(d). $$v(k) = \frac{1}{\gamma^m} v(k_1+i\ell_1,k_2,k_3+i\sigma+i\ell_3) - \sum_{i=1}^{m}\frac{\beta}{\gamma^i} v(k_1+i\ell_1,k_2,k_3+(i-1)\ell_3+i\sigma)$$
$$-\sum_{i=1}^{m} v(k_1+(i+1)\ell_1,k_2,k_3+(i-1)\ell_3+(i-1)\sigma)$$
$$-\sum_{i=1}^{m}\frac{1}{\gamma^{i-1}} v(k_1+i\ell_1,k_2 \pm \ell_2,k_3+(i-1)\ell_3+(i-1)\sigma)$$

$$+\sum_{i-1}^{m}\frac{4\alpha}{\gamma^{i-1}} v(k_1+i\ell_1,k_2,k_3+(i-1)\ell_3+(i-1)\sigma). \quad (19)$$

Proof. The proof is identical to Theorem (2.2).

In most general case, consider homogeneous diffusion medium in $\Re^3$. Let $v(k_1,k_2,k_3,k_4,k_5)$ be the temperature, at position $(k_1,k_2,k_3)$, at time $k_4$ with density (or pressure) $k_5$ and denote $k=(k_1,k_2,k_3,k_4,k_5)$. The partial $\beta-\alpha$ delay heat equation for a homogenous medium is expressed as

$$\underset{\beta(\ell_4,\ell_5)}{\Delta} v(k)=\gamma \underset{\alpha(\pm\ell_{(1,2,3)})}{\Delta} v(k-\sigma), \qquad (20)$$

where $\underset{\alpha(\pm\ell_{(1,2,3)})}{\Delta} = \underset{\alpha(\ell_1)}{\Delta} + \underset{\alpha(-\ell_1)}{\Delta} + \underset{\alpha(\ell_2)}{\Delta} + \underset{\alpha(-\ell_2)}{\Delta} + \underset{\alpha(\ell_3)}{\Delta} + \underset{\alpha(-\ell_3)}{\Delta}$ and $\sigma$ is a delay factor.

As in the case of rod and thin plate, equation $(20)$ has four types of solution as below.

(a). $$v(k)=\frac{1}{\beta^m}v(k-m\ell_{4,5})-\sum_{i=1}^{m}\frac{\gamma}{\beta^i}[v(k+(\pm\ell_1,0,0,\ (i-1)\ell_4-\sigma,(i-1)\ell_5-\sigma)$$
$$+v(k+(0,\pm\ell_2,0,(i-1)\ell_4-\sigma,(i-1)\ell_5-\sigma))$$
$$+v(k+(0,0,\pm\ell_3,(i-1)\ell_4-\sigma,(i-1)\ell_5-\sigma))$$
$$-6\alpha v(k+(0,0,0,\ (i-1)\ell_4-\sigma),(i-1)\ell_5-\sigma] \qquad (21)$$

(b). $$v(k)=\beta^m v(k-m\ell_{4,5})+\sum_{i=1}^{m}\beta^{i-1}\gamma[v(k+(\pm\ell_1,0,0,-i\ell_4-\sigma,-i\ell_5-\sigma))$$
$$+v(k+(0,\pm\ell_2,0,-i\ell_4-\sigma,-i\ell_5-\sigma))+v(k+(0,0,\pm\ell_3,-i\ell_4-\sigma,-i\ell_5-\sigma))$$
$$-6\alpha v(k+(0,0,0,\ -\sigma-i\ell_4,-\sigma-i\ell_5))] \qquad (22)$$

(c). $$v(k)=\frac{1}{\gamma^m}v(k+(-m\ell_1,0,0,\ m\ell_4+m\sigma,m\ell_5+m\sigma))$$
$$-\sum_{i=1}^{m}\frac{\beta}{\gamma^i}v(k+(-i\ell_1,0,0,\ (i-1)\ell_4+i\sigma,(i-1)\ell_5+i\sigma))$$
$$-\sum_{i=1}^{m}\frac{1}{\gamma^{i-1}}v(k+(-(i+1)\ell_1,0,0,\ (i-1)\ell_4+(i-1)\sigma,(i-1)\ell_5+(i-1)\sigma))$$
$$-\sum_{i=0}^{m}\frac{1}{\gamma^{i-1}}v(k+(-i\ell_1,\pm\ell_2,0,(i-1)\ell_4+(i-1)\sigma,(i-1)\ell_5+(i-1)\sigma))$$
$$-\sum_{i-1}^{m}\frac{1}{\gamma^{i-1}}v(k+(-i\ell_1,0,\pm\ell_3,(i-1)\ell_4+(i-1)\sigma,(i-1)\ell_5+(i-1\sigma)))$$
$$-\sum_{i=0}^{m}\frac{6\alpha}{\gamma^{i-1}}v(k+(-i\ell_1,0,0,\ (i-1)\ell_4+(i-1)\sigma,(i-1)\ell_5)+(i-1)\sigma)), \qquad (23)$$

(d). $$v(k)=\frac{1}{\gamma^m}v(k+(\ell_1,0,0,\ m\ell_4+m\sigma,m\ell_5+m\sigma))$$
$$-\sum_{i=1}^{m}\frac{\beta}{\gamma^i}v(k+(i\ell_1,0,0,\ (i-1)\ell_4+i\sigma,(i-1)\ell_5+i\sigma))$$
$$-\sum_{i=1}^{m}\frac{1}{\gamma^{i-1}}v(k+((i+1)\ell_1,0,0,\ (i+1)\ell_4+(i-1)\sigma,(i-1)\ell_5+(i-1)\sigma))$$
$$-\sum_{i=1}^{m}\frac{1}{\gamma^{i-1}}v(k+(i\ell_1,\pm\ell_2,0,(i-1)\ell_4+(i-1)\sigma,(i-1)\ell_5+(i-1)\sigma))$$
$$-\sum_{i=1}^{m}\frac{1}{\gamma^{i-1}}v(k+(i\ell_1,0,\pm\ell_3,(i-1)\sigma+(i-1)\ell_5,(i-1)\sigma))$$
$$-\sum_{i=1}^{m}\frac{6\alpha}{\gamma^{i-1}}v(k+(i\ell_1,0,0,\ (i-1)\ell_4+(i-1)\sigma,(i-1)\ell_4+(i-1)\sigma)). \qquad (24)$$

## IV. Conclusion

The newly developed partial $\beta - \alpha$ difference operator provides relevant results in the field of finite difference methods and heat equation. The nature of propagation of heat through materials of dimensions up to three are derived using partial difference operator. The Theorems (2.2) and (3.2) gives the option for predicting the temperature by knowing the current values at the present time.

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