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Summability Classes of Sequences of Interval Numbers

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Abstract : *In this article we introduce and study the notions of* $\Delta^s_{(v,r)}$ *-lacunary strongly summable,* $\Delta^s_{(v,r)}$ *-Cesàro strongly summable,* $\Delta^s_{(v,r)}$ *- statistically convergent and* $\Delta^s_{(v,r)}$ *-lacunary statistically convergent sequence of interval numbers. Consequently we construct the sequence classes* $\ell^i_\theta\left(\Delta^s_{(v,r)}\right)$, $\sigma^i_1\left(\Delta^s_{(v,r)}\right)$, $s^i\left(\Delta^s_{(v,r)}\right)$ *and* $s^i_\theta\left(\Delta^s_{(v,r)}\right)$ *respectively and investigate the relationship among these classes.*

Keywords: *Sequence of interval numbers; Difference sequence; lacunary strongly summable; Cesàro strongly summable; statistically convergent; lacunary statistically convergent; Completeness.*

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I. Introduction

The concept of interval arithmetic was first suggested by Dwyer [1] in 1951. After developed by Moore [10], Moore and Yang [13]. Furthermore several authors have studied various aspects of the theory and applications of interval numbers in differential equations [13], [14], [15]. The sequence of interval numbers was first introduced by Chiao [20] and defined usual convergence. Bounded and convergence sequences spaces of interval numbers were introduced by Sengonul and Eryilmaz [ ] and showed that these spaces are complete metric space.

A set consisting of closed interval of real numbers x such that $a \le x \le b$ is called an interval number. A real interval can also be considered as a set. Denote the set of all real valued closed intervals by $\mathbb{R}$. Any member of $\mathbb{R}$ is called closed interval and denoted by $\bar{x}$. Thus $\bar{x} = \{x \in \mathbb{R} : a \le x \le b\}$. In [20], an interval number is closed subset of real line $\mathbb{R}$.

Let x_l and x_r be the first and last points of the interval number $\bar{x}$ respectively. For $\bar{x}_1$, $\bar{x}_2 \in \mathbb{R}$, we have

$$\bar{x}_1 = \bar{x}_2 \Leftrightarrow x_{1_l} = x_{2_l}, x_{1r} = x_{2_r}.$$

$$\bar{x}_1 + \bar{x}_2 = \left\{x \in \mathbb{R} : x_{1_l} + x_{2_l} \le x \le x_{1r} + x_{2_r}\right\}$$

$$\alpha\bar{x} = \left\{x \in \mathbb{R} : \alpha x_{1_l} \le x \le \alpha x_{1_r}\right\} \text{ if } \alpha \ge 0.$$

$$= \left\{x \in \mathbb{R} : \alpha x_{1_r} \le x \le \alpha x_{1_l}\right\} \text{ if } \alpha < 0.$$

and

$$\bar{x}_1 . \bar{x}_2 = \left\{x \in \mathbb{R} : \min\left(x_{1_l}.x_{2_l}, x_{1_l}.x_{2_r}, x_{1_r}.x_{2_l}, x_{1_r}.x_{2_r}\right) \le x \le \max\left(x_{1_l}.x_{2_l}, x_{1_l}.x_{2_r}, x_{1_r}.x_{2_l}, x_{1_r}.x_{2_r}\right)\right\}$$

The set of all interval numbers $\mathbb{R}$ is complete metric space under the metric defined by –

$$d\left(\bar{x}, \bar{y}\right) = \max\left\{\left|x_{1_l} - x_{2_l}\right|, \left|x_{1_r} - x_{2_r}\right|\right\} \text{ (see [18]) .}$$

Let us consider the transformation $f : \mathbb{N} \to \mathbb{R}$ by $k \to f(k) = \bar{x}$ where $\bar{x} = (\bar{x}_k)$ which is known as sequence of interval numbers. $\bar{x}_k$ denotes the k^{th} term of the sequence $\bar{x} = (\bar{x}_k)$. The set of all sequences of interval numbers is denoted by w^i can be found in [18].

II. Definitions and Main Results

Let X be a linear metric space. A function $p : X \to R$ is called paranorm if –

(1) $p(x) \geq 0$ for all $x \in X$

(2) $p(-x) = p(x)$ for all $x \in X$

(3) $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$

(4) If (λ_n) be a sequence of scalars such that $\lambda_n \to 0$ as $n \to \infty$ and (x_n) be a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0 \Rightarrow x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space.

Let $\phi = (\phi_n)_n$ be a sequence of Young functions i.e. $\phi_n : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing and convex function such that $\phi_n(x) = 0$ for $x > 0$ and $\phi_n(0) = 0$. The Musielak-Orlicz sequence space ℓ^ϕ is given by –

$$\ell^\phi = \left\{ x = (x_n)_n : \sum_n \phi_n\left(\lambda |x_n|\right) < \infty, \lambda > 0 \right\}$$

.This becomes Banach space under the norm(Luxemburg)

$$|x|_\phi = \inf \left\{ \eta > 0 : \sum_n \phi_n\left(\frac{|x_n|}{\eta}\right) \leq 1, \eta > 0 \right\}$$

Let $\phi = (\phi_k)$ be the sequence of Young functions. The space consisting of all those sequences $\bar{x} = (\bar{x}_k)$ in w^i such that

$\phi\left(\dfrac{|\bar{x}_k|^{\frac{1}{k}}}{\eta}\right) \to 0$ as $k \to \infty$ for some $\eta > 0$ is known as class of entire sequences of interval numbers defined by sequence of Young functions and is denoted by $\bar{\Gamma}_\phi$. The space consisting of all those sequences $\bar{x} = (\bar{x}_k)$ in w^i such that $\sup_k \left(\phi\left(\dfrac{|\bar{x}_k|^{\frac{1}{k}}}{\eta}\right)\right) < \infty$ for some $\eta > 0$ is known as class of analytic sequences of interval numbers defined by sequence of Young functions and is denoted by $\bar{\Lambda}_\phi$.

Lemma 2.1: Let (α_k) and (β_k) be sequences of real or complex numbers and (p_k) be a bounded sequence of positive real numbers, then

$$|\alpha_k + \beta_k|^{p_k} \leq D(|\alpha_k|^{p_k} + |\beta_k|^{p_k})$$

and $|\lambda|^{p_k} \leq \max(1, |\lambda|^H)$

where $D = \max(1, |\lambda|^{H-1}), H = \sup p_k$, λ is any real or complex number.

Lemma 2.2: If d is translation invariant then

(a) $d\left(\bar{x}_k+\bar{y}_k,\bar{0}\right)\le d\left(\bar{x}_k,\bar{0}\right)+d\left(\bar{y}_k,\bar{0}\right)$

(b) $d\left(\alpha\bar{x}_k,\bar{0}\right)\le|\alpha|d\left(\bar{x}_k,\bar{0}\right)$, $|\alpha|>1$.

Let $\bar{x}=\left(\bar{x}_k\right)$ be sequence of interval numbers, $p=\left(p_k\right)$ be sequence of strictly positive integers, $A=\left(a_{nk}\right)$ be non negative regular matrix and $\phi=\left(\phi_k\right)$ be a sequence of Young functions, we define the following classes of sequences of interval numbers as follows:

$$\bar{\Gamma}_\phi(A,p)=\left\{\bar{x}=\left(\bar{x}_k\right):\lim_{k\to\infty}\sum_k a_{nk}\left[d\left(\phi\left(\frac{\left|\bar{x}_k\right|^{1/k}}{\eta},0\right)\right)\right]^{p_k}=0\right\}$$

$$\bar{\Lambda}_\phi(A,p)=\left\{\bar{x}=\left(\bar{x}_k\right):\sup_n\left(\sum_k a_{nk}\left[d\left(\phi\left(\frac{\left|\bar{x}_k\right|^{1/k}}{\eta},0\right)\right)\right]^{p_k}\right)<\infty\right\}$$

for some $\eta>0$. We can specialize these spaces as follows:

(a) If $A=I$, the unit matrix then –

$$\bar{\Gamma}_\phi(I,p)=\left\{\bar{x}=\left(\bar{x}_k\right):\lim_{k\to\infty}\left[d\left(\phi\left(\frac{\left|\bar{x}_k\right|^{1/k}}{\eta},0\right)\right)\right]^{p_k}=0\right\}$$

$$\bar{\Lambda}_\phi(I,p)=\left\{\bar{x}=\left(\bar{x}_k\right):\sup_k\left(\left[d\left(\phi\left(\frac{\left|\bar{x}_k\right|^{1/k}}{\eta},0\right)\right)\right]^{p_k}\right)<\infty\right\}$$

(b) If we take $\phi(x)=x$ then we get –

$$\bar{\Gamma}(A,p)=\left\{\bar{x}=\left(\bar{x}_k\right):\lim_{k\to\infty}\sum_k a_{nk}\left[d\left(\frac{\left|\bar{x}_k\right|^{1/k}}{\eta},0\right)\right]^{p_k}=0\right\}$$

$$\bar{\Lambda}(A,p)=\left\{\bar{x}=\left(\bar{x}_k\right):\sup_n\left(\sum_k a_{nk}\left[d\left(\frac{\left|\bar{x}_k\right|^{1/k}}{\eta},0\right)\right]^{p_k}\right)<\infty\right\}$$

(c) If $A=\left(a_{nk}\right)$ is Cesaro matrix of order 1 and $p_k=p$ then we have -

$$\bar{\Gamma}_\phi(p)=\left\{\bar{x}=\left(\bar{x}_k\right):\lim_{k\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left[d\left(\phi\left(\frac{\left|\bar{x}_k\right|^{1/k}}{\eta},0\right)\right)\right]^{p}\right\}=0$$

$$\overline{\Lambda}_\phi(p)=\left\{\overline{x}=\left(\overline{x}_k\right):\sup_n\left(\frac{1}{n}\sum_{k=1}^{n}\left[d\left(\phi\left(\frac{\left|\overline{x}_k\right|^{1/k}}{\eta},0\right)\right)\right]^p\right)<\infty\right\}$$

The space $\overline{\Gamma}$ is defined as follows ;

$$\overline{\Gamma}=\left\{\overline{x}=\left(\overline{x}_k\right):\lim_{k\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{\left|\overline{x}_k\right|^{1/k}}{\eta}=0\right\}\text{ for some }\eta>0.$$

III. Main Results

Theorem 3.1: If d is translation invariant then the class of sequence $\overline{\Gamma}_\phi(p)$ is closed under addition and scalar multiplication of interval numbers.

Proof: Let $\overline{x}=\left(\overline{x}_k\right)\in\overline{\Gamma}_\phi(p)$ and $\overline{y}=\left(\overline{y}_k\right)\in\overline{\Gamma}_\phi(p)$

In order to prove the result, we need to find some $\eta_3>0$ such that

$$\sum_{k=1}^{n}\frac{1}{n}\left[d\left(\phi\left(\frac{\left|\left(a\overline{x}_k+b\overline{x}_k\right)\right|^{1/k}}{\eta_3},0\right)\right)\right]^p\to 0\quad\text{as }k\to\infty$$

Since $\overline{x}=\left(\overline{x}_k\right)\in\overline{\Gamma}_\phi(p)$ and $\overline{y}=\left(\overline{y}_k\right)\in\overline{\Gamma}_\phi(p)$, there exists some $\eta_1>0$ and $\eta_2>0$ such that –

$$\sum_{k=1}^{n}\frac{1}{n}\left[d\left(\phi\left(\frac{\left|\overline{x}_k\right|^{1/k}}{\eta_1},0\right)\right)\right]^p\to 0\quad\text{as }k\to\infty\quad\text{and}$$

$$\sum_{k=1}^{n}\frac{1}{n}\left[d\left(\phi\left(\frac{\left|\overline{y}_k\right|^{1/k}}{\eta_2},0\right)\right)\right]^p\to 0\quad\text{as }k\to\infty.$$

Since ϕ is non-decreasing, we have

$$\sum_{k=1}^{n}\frac{1}{n}\left[d\left(\phi\left(\frac{\left|\left(a\overline{x}_k+b\overline{y}_k\right)\right|^{1/k}}{\eta_3},0\right)\right)\right]^p\le\sum_{k=1}^{n}\frac{1}{n}\left[d\left(\phi\left(\frac{\left|\left(a\overline{x}_k\right)\right|^{1/k}}{\eta_3}+\frac{\left|\left(b\overline{y}_k\right)\right|^{1/k}}{\eta_3},0\right)\right)\right]^p$$

$$\le\sum_{k=1}^{n}\frac{1}{n}\left[d\left(\phi\left(\frac{|a|^{1/k}\left|\overline{x}_k\right|^{1/k}}{\eta_3}+\frac{|b|^{1/k}\left|\overline{y}_k\right|^{1/k}}{\eta_3},0\right)\right)\right]^p$$

$$\le\sum_{k=1}^{n}\frac{1}{n}\left[d\left(\phi\left(\frac{|a|\left|\overline{x}_k\right|^{1/k}}{\eta_3}+\frac{|b|\left|\overline{y}_k\right|^{1/k}}{\eta_3},0\right)\right)\right]^p$$

Take η_3 such that

$$\frac{1}{\eta_3} = \min\left\{\frac{1}{|a|^p}\frac{1}{\eta_1}, \frac{1}{|b|^p}\frac{1}{\eta_2}\right\}$$

Then,

$$\sum_{k=1}^{n}\frac{1}{n}\left[d\left(\phi\left(\frac{\left|\left(a\bar{x}_k + b\bar{y}_k\right)\right|^{1/k}}{\eta_3}, 0\right)\right)\right]^p \le \sum_{k=1}^{n}\frac{1}{n}\left[d\left(\phi\left(\frac{\left|\bar{x}_k\right|^{1/k}}{\eta_1} + \frac{\left|\bar{y}_k\right|^{1/k}}{\eta_2}, 0\right)\right)\right]^p$$

$$\le \sum_{k=1}^{n}\frac{1}{n}\left[d\left(\phi\left(\frac{\left|\bar{x}_k\right|^{1/k}}{\eta_1}, 0\right)\right)\right]^p + \sum_{k=1}^{n}\frac{1}{n}\left[d\left(\phi\left(\frac{\left|\bar{y}_k\right|^{1/k}}{\eta_2}, 0\right)\right)\right]^p$$

Hence $\displaystyle\sum_{k=1}^{n}\frac{1}{n}\left[d\left(\phi\left(\frac{\left|\left(a\bar{x}_k + b\bar{y}_k\right)\right|^{1/k}}{\eta_3}, 0\right)\right)\right]^p \to 0$ as $k \to \infty$.

So $a\bar{x}_k + b\bar{x}_k \in \bar{\Gamma}_\phi(p)$. This completes the proof.

Theorem 3.2. The class of sequence $\bar{\Gamma}_\phi(p)$ is a complete metric space under the metric $'h'$ defined by –

$$h\left(\bar{x}, \bar{y}\right) = \sup_n\left[\frac{1}{n}\sum_{k=1}^{n} d\left(\phi\left(\frac{\left|\bar{x}_k - \bar{y}_k\right|^{1/k}}{\eta}, 0\right)\right)\right]^p$$

Proof. Let $\left\{\bar{x}^{(i)}\right\}$ be Cauchy sequence in $\bar{\Gamma}_\phi(p)$.

Then for any given $\varepsilon > 0$ there exists a positive integer n_1 such that

$$h\left(\bar{x}^{(i)}, \bar{y}^{(j)}\right) < \varepsilon \qquad \text{for all} \quad i, j \ge n_1.$$

Therefore

$$\sup_n\left[\frac{1}{n}\sum_{k=1}^{n} d\left(\phi\left(\frac{\left|\bar{x}_k^{(i)} - \bar{y}_k^{(j)}\right|^{1/k}}{\eta}, 0\right)\right)\right]^p < \varepsilon \quad \text{for all} \quad i, j \ge n_1.$$

Consequently $\left\{\bar{x}_k^{(i)}\right\}$ is a Cauchy sequence in the metric space of interval numbers which is complete and so $\bar{x}_k^{(i)} \to \bar{x}_k$ as $i \to \infty$.

Once can find that -

$$\left[\frac{1}{n}\sum_{k=1}^{n} d\left(\phi\left(\frac{\left|\bar{x}_k^{(i)} - \bar{x}_k\right|^{1/k}}{\eta}, 0\right)\right)\right]^p < \varepsilon, \quad i \ge n_1.$$

Now,

$$\left[\frac{1}{n}\sum_{k=1}^{n} d\left(\phi\left(\frac{\left|\bar{x}_k\right|^{1/k}}{\eta},0\right)\right)\right]^p \le \left[\frac{1}{n}\sum_{k=1}^{n} d\left(\phi\left(\frac{\left|\bar{x}_k - \bar{x}_k^{(n_1)}\right|^{1/k}}{\eta},0\right)\right)\right]^p + \left[\frac{1}{n}\sum_{k=1}^{n} d\left(\phi\left(\frac{\left|\bar{x}_k^{n_1}\right|^{1/k}}{\eta},0\right)\right)\right]^p$$

$< \varepsilon + 0$ as $n \to \infty$.

Thus $\left[\frac{1}{n}\sum_{k=1}^{n} d\left(\phi\left(\frac{\left|\bar{x}_k\right|^{1/k}}{\eta},0\right)\right)\right]^p < \varepsilon$

and so $\left(\bar{x}_k\right) \in \bar{\Gamma}_\phi\left(p\right)$.

Hence $\bar{\Gamma}_\phi\left(p\right)$ is a complete metric space. This completes the proof.

Theorem 3.3. Let $\bar{x} = \left(\bar{x}_k\right)$ be sequence of interval numbers. The sequence class $\bar{\Gamma}_\phi\left(A, p\right)$ is complete w.r.t the topology generated by the paranorm h defined by –

$$h\left(\bar{x}\right) = \sup_k \left(\sum_{k=1}^{n} a_{nk}\left[d\left(\phi\left(\frac{\left|\bar{x}_k\right|^{1/k}}{\eta},0\right)\right)\right]^{p_k}\right)^{\frac{1}{M}}$$

Where $M = \max\left\{1, \sup_k\left(\frac{p_k}{M}\right)\right\}$.

Proof. Obviously $h\left(\theta\right) = 0$ and $h\left(-\bar{x}\right) = h\left(\bar{x}\right)$. It can also be easily seen that

$h\left(\bar{x}+\bar{y}\right) \le h\left(\bar{x}\right) + h\left(\bar{y}\right)$ as d is translation invariant.

Now for any scalar λ, we have $\left|\lambda\right|^{p_k/M} < \max\left(1, \sup\left|\lambda\right|\right)$, so that

$h\left(\lambda\bar{x}\right) < \max\left(1, \sup\left|\lambda\right|\right)$, λ fixed implies $\lambda\bar{x} \to \theta$. Now let $\lambda \to \theta$, $\bar{x}$ fixed for $\sup\left|\lambda\right| < 1$, we have

$$\left(\sum_{k=1}^{n} a_{nk}\left[d\left(\phi\left(\frac{\left|\bar{x}_k\right|^{1/k}}{\eta},0\right)\right)\right]^{p_k}\right)^{\frac{1}{M}} < \varepsilon \text{ for some } N > N\left(\varepsilon\right).$$

Also for $1 \le n \le N$ and $\left(\sum_{k=1}^{n} a_{nk}\left[d\left(\phi\left(\frac{\left|\bar{x}_k\right|^{1/k}}{\eta},0\right)\right)\right]^{p_k}\right)^{\frac{1}{M}} < \varepsilon$ there exists m such that

$$\left(\sum_{k=m}^{n} a_{nk}\left[d\left(\phi\left(\frac{\left|\lambda\bar{x}_k\right|^{1/k}}{\eta},0\right)\right)\right]^{p_k}\right)^{\frac{1}{M}} < \varepsilon.$$

Taking λ small enough, we then find

$$\left(\sum_{k=m}^{n} a_{nk}\left[d\left(\phi\left(\frac{\left|\lambda \bar{x}_k\right|^{1/k}}{\eta},0\right)\right)\right]^{p_k}\right)^{\frac{1}{M}} < 2\varepsilon \text{ for all } k.$$

Hence $h\left(\lambda \bar{x}\right) \to 0$ as $\lambda \to 0$. So h is a paranorm on $\bar{\Gamma}_\phi\left(A,p\right)$.

To show the completeness, let $\left\{\bar{x}^{(i)}\right\}$ be Cauchy sequence in $\bar{\Gamma}_\phi\left(A,p\right)$.

Then for given $\varepsilon > 0$ there exists positive integer r such that –

$$\left(\sum a_{nk}\left[d\left(\phi\left(\frac{\left|\bar{x}_k^i - \bar{x}_k^j\right|^{1/k}}{\eta},0\right)\right)\right]^{p_k}\right)^{\frac{1}{M}} < \varepsilon \text{ for all } j \to \infty\ i, j \geq r.$$

Since d is translation invariant, so

$$\left(\sum a_{nk}\left[d\left(\phi\left(\frac{\left|\bar{x}_k^i - \bar{x}_k^j\right|^{1/k}}{\eta},0\right)\right)\right]^{p_k}\right)^{\frac{1}{M}} < \varepsilon \text{ for all } \quad i, j \geq r. \text{ and each n.}$$

Hence

$$\left[d\left(\phi\left(\frac{\left|\bar{x}_k^i - \bar{x}_k^j\right|^{1/k}}{\eta},0\right)\right)\right] < \varepsilon \text{ for all } \quad i, j \geq r.$$

Therefore $\left\{\bar{x}^{(i)}\right\}$ is a Cauchy sequence in the metric space of interval numbers which is complete and hence $\bar{x}^{(j)} \to \bar{x}$ as $j \to \infty$

Keeping $r_0 \geq r$ and letting $j \to \infty$, once can find that –

$$\left(\sum a_{nk}\left[d\left(\phi\left(\frac{\left|\bar{x}_k^i - \bar{x}_k\right|^{1/k}}{\eta},0\right)\right)\right]\right) < \varepsilon \text{ for all } \quad r_0 \geq r.$$

Since d is translation invariant, therefore

$$\left(\sum a_{nk}\left[d\left(\phi\left(\frac{\left|\bar{x}_k^i - \bar{x}_k\right|^{1/k}}{\eta},0\right)\right)\right]^{p_k}\right)^{1/M} < \varepsilon$$

i.e $\bar{x}^{(i)} \to \bar{x}$ in $\bar{\Gamma}_\phi\left(A,p\right)$. It can be easily seen that $\bar{x} \in \bar{\Gamma}_\phi\left(A,p\right).$

Thus $\bar{\Gamma}_\phi\left(A,p\right)$ is complete. This completes the proof.

Theorem 3.4. If $0 < \inf p_k \le p_k \le 1$, then $\overline{\Gamma}_{\phi}(A,p) \subset \overline{\Gamma}_{\phi}(A)$.

Proof. Let $\bar{x} = (\bar{x}_k) \in \overline{\Gamma}_{\phi}(A,p)$. Since $0 < \inf p_k \le p_k \le 1$, the result follows from the following inequality

$$\sum_k a_{nk}\left[d\left(\phi\left(\frac{|\bar{x}_k|^{1/k}}{\eta},0\right)\right)\right] \le \sum_k a_{nk}\left[d\left(\phi\left(\frac{|\bar{x}_k|^{1/k}}{\eta},0\right)\right)\right]^{p_k}$$

Theorem 3.5. If $1 \le p_k \le \sup p_k < \infty$, then $\overline{\Gamma}_{\phi}(A) \subset \overline{\Gamma}_{\phi}(A,p)$..

Proof. $\bar{x} = (\bar{x}_k) \in \overline{\Gamma}_{\phi}(A)$. Since $1 \le p_k \le \sup p_k < \infty$ then for each $0 < \varepsilon < 1$ there exist a positive integer n_0 such that

$$\sum_k a_{nk}\left[d\left(\phi\left(\frac{|\bar{x}_k|^{1/k}}{\eta},0\right)\right)\right] \le \varepsilon < 1 \text{ for some } n \ge n_0.$$

The result follows from the following inequality

$$\sum_k a_{nk}\left[d\left(\phi\left(\frac{|\bar{x}_k|^{1/k}}{\eta},0\right)\right)\right]^{p_k} \le \sum_k a_{nk}\left[d\left(\phi\left(\frac{|\bar{x}_k|^{1/k}}{\eta},0\right)\right)\right].$$

Theorem 3.6. Suppose $\bar{x} = (\overline{x_k})$ is strongly $\Delta^s_{(v,r)}$ -lacunary strongly summable to X_0. Then

$$\lim_{p\to\infty} \frac{1}{h_p} \sum_{k\in I_p} d\left(\Delta^s_{(v,r)}\bar{x}_k, \bar{x}_0\right) = 0.$$

Now the result follows from the following inequality:

$$\sum_{k\in I_p} d\left(\Delta^s_{(v,r)}\bar{x}_k, \bar{x}_0\right) \ge \varepsilon \operatorname{card}\left\{k \le n : d(\Delta^s_{(v,r)}\bar{x}_k, \bar{x}_0) \ge \varepsilon\right\}$$

Theorem 3.7. If a sequence $\bar{x} = (\overline{x_k})$ of interval numbers is $\Delta^s_{(v,r)}$ -bounded and $\Delta^s_{(v,r)}$ - statistically convergent, then it is $\Delta^s_{(v,r)}$ - Cesàro strongly summable.

Proof. Suppose $\bar{x} = (\overline{x_k})$ is $\Delta^s_{(v,r)}$ -bounded and $\Delta^s_{(v,r)}$ - statistically convergent to $\bar{x}_0$. Since $\bar{x} = (\overline{x_k})$ is $\Delta^s_{(v,r)}$ -bounded, we can find a interval number M such that

$$d(\Delta^s_{(v,r)}\bar{x}_k, \bar{x}_0) \le M \quad \text{for all } k \in \mathbb{N}$$

Again since $\bar{x} = (\overline{x_k})$ is $\Delta^s_{(v,r)}$ - statistically convergent to $\bar{x}_0$, for every $\varepsilon > 0$

$$\lim_n \frac{1}{n} \operatorname{card}\left\{k \le n : d(\Delta^s_{(v,r)}\bar{x}_k, \bar{x}_0) \ge \varepsilon\right\} = 0,$$

Now the result follows from the following inequality

$$\frac{1}{n}\sum_{1\le k\le n} d\left(\Delta^s_{(v,r)}\bar{x}_k, \bar{x}_0\right) =$$

$$\frac{1}{n}\sum_{\substack{1\le k\le n\\ d(\Delta^s_{(v,r)}\bar{x}_k,\bar{x}_0)\ge\varepsilon}} d(\Delta^s_{(v,r)}\bar{x}_k,\bar{x}_0)+\frac{1}{n}\sum_{\substack{1\le k\le n\\ d(\Delta^s_{(v,r)}\bar{x}_k,\bar{x}_0)<\varepsilon}} d(\Delta^s_{(v,r)}\bar{x}_k,\bar{x}_0)$$

$$\le\frac{M}{n}\operatorname{card}\left\{k\le n: d(\Delta^s_{(v,r)}\bar{x}_k,\bar{x}_0)\ge\varepsilon\right\}+\varepsilon$$

Theorem 3.8. Let θ be a lacunary sequence. Then if a sequence $\bar{x}=\left(\overline{x_k}\right)$ is $\Delta^s_{(v,r)}$ -bounded and $\Delta^s_{(v,r)}$ - lacunary statistically convergent, then it is $\Delta^s_{(v,r)}$ - lacunary strongly summable.

Proof. Proof follows by similar arguments as applied to prove above Theorem.

Theorem 3.9. Let θ be a lacunary sequence and $\bar{x}=\left(\overline{x_k}\right)$ be $\Delta^s_{(v,r)}$ -bounded. Then X is $\Delta^s_{(v,r)}$ - lacunary statistically convergent if and only if it is $\Delta^s_{(v,r)}$ - lacunary strongly summable.

Proof. Proof follows by combining the Theorems 3.1 and 3.3.

Theorem 3.10. If a sequence $\bar{x}=\left(\overline{x_k}\right)$ is $\Delta^s_{(v,r)}$ - statistically convergent and lim $\inf_p\left(\frac{h_p}{p}\right)>0$ then it is $\Delta^s_{(v,r)}$ -lacunary statistically convergent.

Proof. Assume the given conditions. For a given ε > 0, we have

$$\left\{k\in I_p: d(\Delta^s_{(v,r)}\bar{x}_k,\bar{x}_0)\ge\varepsilon\right\}\subset\left\{k\le n: d(\Delta^s_{(v,r)}\bar{x}_k,\bar{x}_0)\ge\varepsilon\right\}$$

Hence the proof follows from the following inequality:

$$\frac{1}{p}\operatorname{card}\left\{k\le p: d(\Delta^s_{(v,r)}\bar{x}_k,\bar{x}_0)\ge\varepsilon\right\}\ge\frac{1}{p}\operatorname{card}\left\{k\in I_p: d(\Delta^s_{(v,r)}\bar{x}_k,\bar{x}_0)\ge\varepsilon\right\}$$

$$=\frac{h_p}{p}\frac{1}{h_p}\operatorname{card}\left\{k\in I_p: d(\Delta^s_{(v,r)}\bar{x}_k,\bar{x}_0)\ge\varepsilon\right\}$$

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