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On Certain Special Finsler Spaces of Dimension Five

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***Abstract**: L. Berwald [1-3] developed the theory of two dimensional Finsler spaces using a special orthonormal frame. His study was based on two scalars I and R, which were scalar components of the torsion tensor and the curvature tensor with respect to orthonormal frame. A. Moor [4] developed the theory of three dimensional Finsler spaces using a special orthonormal frame whose first vector was the normalized supporting element and the second vector was the normalized torsion tensor. He found three main scalars which are regarded as a generalization of Berwald main scalar I. Makoto Matsumoto [5] also studied three dimensional Finsler spaces using Moor's orthonormal frame. He obtained some results for certain Finsler spaces. T. N. Pandey and D. K. Diwedi [6] extended this idea to four dimensional Finsler spaces. P. N. Pandey and M. K. Gupta [7] discussed certain special Finsler spaces of dimension four and obtained some results for such spaces. Gauree Shankar, G. C. Chaubey and Vinay Pandey [8] discussed five dimensional Finsler space on the basis of Miron frame and find some interesting results. In this paper, we have discussed certain special Finsler spaces of dimension five and obtained some results for such spaces.*

***Keywords**: Finsler space, Miron frame, Berwald space, Landsberg space. 2000 Math Subject Classification: 53B40.*

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I. Introduction

Consider a five-dimensional Finsler space F^5 with the fundamental function $L(x, y)$, Gauree Shankar, G. C. Chaubey and Vinay Pandey [8] have constructed the orthonormal frame on the basis of Moor's frame. In this frame an arbitrary tensor is expressed in terms of scalar components as follows:

$$T^i_{jk} = T_{\alpha\beta\gamma}\, e^i_{\alpha)} e_{\beta)j} e_{\gamma)k} \tag{1.1}$$

where $e^i_{1)} = l^i$, $e^i_{2)} = m^i$, $e^i_{3)} = n^i$, $e^i_{4)} = p^i$ and $e^i_{5)} = q^i$ and the summation convension is applied to Greek indices also.

The scalar components $T_{\alpha\beta\gamma}$ are given by

$$T_{\alpha\beta\gamma} = T^i_{jk}\, e_{\alpha)i} e^j_{\beta)} e^k_{\gamma)}.$$

Let $H_{\alpha)\beta\gamma}$ and $\frac{1}{L} V_{\alpha)\beta\gamma}$ respectively be scalar components of the $h-$ and $v-$covariant derivatives $e^i_{\alpha)|j}$ and $e^i_{\alpha)}|_j$ of the vector $e^i_{\alpha)}$, i.e.,

$$e^i_{\alpha)|j} = H_{\alpha)\beta\gamma}\, e^i_{\beta)}\, e_{\gamma)j}, \tag{1.2}$$

and

$$L\, e^i_{\alpha)}|_j = V_{\alpha)\beta\gamma}\, e^i_{\beta)}\, e_{\gamma)j}, \tag{1.3}$$

The orthogonality of Miron frame yields

$$H_{1)\beta\gamma} = 0, \qquad H_{\alpha)\beta\gamma} = -H_{\beta)\alpha\gamma},$$

$$V_{1)\beta\gamma} = \delta_{\beta\gamma} - \delta_{1\beta}\delta_{1\gamma}, \qquad V_{\alpha)\beta\gamma} = -V_{\beta)\alpha\gamma}.$$

We now define Finsler vector fields:

$$\begin{aligned} H_{2)3\beta}e_{\beta)i} &= h_i = h_\beta e_{\beta)i} \\ H_{2)4\beta}e_{\beta)i} &= J_i = J_\beta e_{\beta)i} \\ H_{2)5\beta}e_{\beta)i} &= k_i = k_\beta e_{\beta)i} \\ H_{3)4\beta}e_{\beta)i} &= h_i' = h_\beta' e_{\beta)i} \\ H_{3)5\beta}e_{\beta)i} &= J_i' = J_\beta' e_{\beta)i} \\ H_{4)5\beta}e_{\beta)i} &= k_i' = k_\beta' e_{\beta)i} \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} V_{2)3\gamma}e_{\gamma)i} &= u_i = u_{\gamma)}e_{\gamma)i} \\ V_{2)4\gamma}e_{\gamma)i} &= v_i = v_{\gamma)}e_{\gamma)i} \\ V_{2)5\gamma}e_{\gamma)i} &= w_i = w_{\gamma)}e_{\gamma)i} \\ V_{3)4\gamma}e_{\gamma)i} &= u_i' = u_{\gamma)}' e_{\gamma)i} \\ V_{3)5\gamma}e_{\gamma)i} &= v_i' = v_{\gamma)}' e_{\gamma)i} \\ V_{4)5\gamma}e_{\gamma)i} &= w_i' = w_{\gamma)}' e_{\gamma)i}. \end{aligned} \tag{1.5}$$

Definition : The Finsler vector fields $(h_i, J_i, k_i, h_i', J_i', k_i')$ and $(u_i, v_i, w_i, u_i', v_i', w_i')$ are called $h-$ and $v-$connection vectors respectively.

The skew symmetric matrices $H_{\alpha)\beta\gamma}$ and $V_{\alpha)\beta\gamma}$ for fixed γ are given by

$$H_{\alpha)\beta\gamma} = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & h_\gamma & J_\gamma & k_\gamma \\ 0 & -h_\gamma & 0 & h_\gamma' & J_\gamma' \\ 0 & -J_\gamma & -h_\gamma' & 0 & k_\gamma' \\ 0 & -k_\gamma & -J_\gamma' & -k_\gamma' & 0 \end{vmatrix}$$

and

$$V_{\alpha)\beta\gamma} = \begin{vmatrix} 0 & \delta_{2\gamma} & \delta_{3\gamma} & \delta_{4\gamma} & \delta_{5\gamma} \\ -\delta_{2\gamma} & 0 & u_\gamma & v_\gamma & w_\gamma \\ -\delta_{3\gamma} & -u_\gamma & 0 & u_\gamma' & v_\gamma' \\ -\delta_{4\gamma} & -v_\gamma & -u_\gamma' & 0 & w_\gamma' \\ -\delta_{5\gamma} & -w_\gamma & -v_\gamma' & -w_\gamma' & 0 \end{vmatrix}.$$

Taking $h-$covariant derivative of (1.1), we get

$$\begin{aligned} T^i_{jk|h} = {} & (\delta_h T_{\alpha\beta\gamma})e^i_{\alpha)}e_{\beta)j}e_{\gamma)k} + T_{\alpha\beta\gamma}e^i_{\alpha)|h}e_{\beta)j}e_{\gamma)k} + T_{\alpha\beta\gamma}e^i_{\alpha)}e_{\beta)j|h}e_{\gamma)k} \\ & + T_{\alpha\beta\gamma}e^i_{\alpha)}e_{\beta)j}e_{\gamma)k|h}. \end{aligned} \tag{1.6}$$

Let $T_{\alpha\beta\gamma,\delta}$ be scalar components of $T^i_{jk|h}$ then

$$T^i_{jk|h} = T_{\alpha\beta\gamma,\delta}\, e^i_{\alpha)}e_{\beta)j}e_{\gamma)k}e_{\delta)h}. \tag{1.7}$$

Then we obtain

$$T_{\alpha\beta\gamma,\delta} = (\delta_h T_{\alpha\beta\gamma})e^h_{\delta)} + T_{\mu\beta\gamma}H_{\mu)\alpha\delta} + T_{\alpha\mu\gamma}H_{\mu)\beta\delta} + T_{\alpha\beta\mu}H_{\mu)\gamma\delta}. \tag{1.8}$$

II. Main Scalars of a Five Dimensional Finsler Space

Matsumoto [5] have shown that if $C_{\alpha\beta\gamma}$ are the scalar components of LC_{ijk} with respect to the Miron frame, then

(i) $C_{\alpha\beta\gamma}$ are completely symmetric,

(ii) $C_{1\beta\gamma} = 0$,

(iii) $C_{2\mu\mu} = LC$, $C_{3\mu\mu} = C_{4\mu\mu} = \ldots = C_{n\mu\mu} = 0$ for $n \geq 3$ where C is the length of C^i .

Therefore in five dimensional Finsler space, we have

$$\begin{aligned} &C_{1\beta\gamma} = 0, \\ &C_{222} + C_{233} + C_{244} + C_{255} = LC, \\ &C_{422} + C_{433} + C_{444} + C_{455} = 0, \\ &C_{522} + C_{533} + C_{544} + C_{555} = 0. \end{aligned} \tag{2.1}$$

Thus putting

$$\begin{array}{llll} C_{222} = H, & C_{233} = I, & C_{244} = K, & C_{255} = M, \\ C_{333} = J, & C_{344} = J', & C_{444} = H', & C_{334} = I', \\ C_{234} = K', & C_{355} = J'', & C_{455} = M', & C_{555} = H'' \\ C_{335} = I'', & C_{445} = K'', & C_{235} = N, & C_{245} = N', \\ C_{345} = M'', & & & \end{array} \tag{2.2}$$

we have,

$$C_{322} = -(J + J' + J''), \quad C_{224} = -(H' + I' + M'), \quad C_{225} = -(H'' + I'' + M'').$$

Definition. The seventeen scalars $H, I, J, K, H', I', J', K', H'', I'', J'', K'', M, M', M'', N, N'$ are called the main scalars of a five-dimensional Finsler space.

III. Certain Theorems

If $C_{\alpha\beta\gamma}$ be the scalar components of LC_{ijk} , then from (1.7), we have

$$LC_{hij|k} = C_{\alpha\beta\gamma,\delta}\, e_{\alpha)h} e_{\beta)i} e_{\gamma)j} e_{\delta)k} \tag{3.1}$$

where

$$C_{\alpha\beta\gamma,\delta} = (\delta_k C_{\alpha\beta\gamma}) e^k_{\delta)} + C_{\mu\beta\gamma} H_{\mu)\alpha\delta} + C_{\alpha\mu\gamma} H_{\mu)\beta\delta} + C_{\alpha\beta\mu} H_{\mu)\gamma\delta}. \tag{3.2}$$

The explicit form of $C_{\alpha\beta\gamma,\delta}$ is obtained as follows:

(a) $C_{1\beta\gamma,\delta} = 0,$

(b) $C_{222,\delta} = H_{,\delta} + 3(J + J' + J'')h_\delta + 3(H' + I' + M')J_\delta + 3(H'' + I'' + K'')k_\delta,$

(c) $$\begin{aligned} C_{223,\delta} = &-(J + J' + J'')_{,\delta} + (H - 2I)h_\delta - 2K'J_\delta - 2Nk_\delta \\ &+ (H' + I' + M')h'_\delta + (H'' + I'' + M'')J'_\delta, \end{aligned}$$

(d) $$\begin{aligned} C_{224,\delta} = &-(H' + I' + M')_{,\delta} - 2K'h_\delta + (H - 2K)J_\delta - 2N'k_\delta \\ &- (J + J' + J'')h'_\delta + (H'' + I'' + K'')k'_\delta, \end{aligned}$$

(e) $$\begin{aligned} C_{225,\delta} = &-(H'' + I'' + K'')_{,\delta} - 2Nh_\delta - 2N'J_\delta + (H - 2M)k_\delta \\ &- (J + J' + J''J'_\delta - (H' + I' + M')k'_\delta, \end{aligned}$$

(f) $C_{233,\delta} = I_{,\delta} - (3J + 2J' + 2J'')h_\delta - I'J_\delta - I''k_\delta - 2NJ'_\delta - 2K'h'_\delta,$

(g) $$\begin{aligned} C_{234,\delta} = &K'_{,\delta} - (2I' + H' + M')h_\delta - (2J' + J + J'')J_\delta - M''k_\delta \\ &- (K - I)h'_\delta - N'J'_\delta - Nk'_\delta, \end{aligned}$$

(h) $C_{235,\delta} = N_{,\delta} - (2I'' + H'' + K'')h_\delta - M''J_\delta - (J + J' + 2J'')k_\delta$

$$-N'h'_\delta - (M - I)J'_\delta + K'k'_\delta,$$

(i) $$C_{244,\delta} = K_{,\delta} - J'h_\delta - (3H' + 2I' + 2M')J_\delta + 2Kh'_\delta - K''k_\delta - 2N'k'_\delta,$$

(j) $$C_{245,\delta} = N'_{,\delta} - M''h_\delta - (H'' + I'' + 2K'')J_\delta + Nh'_\delta - (H' + I' + 2M')k_\delta + K'J'_\delta + (K - M)k'_\delta,$$

(k) $$C_{255,\delta} = M_{,\delta} - J''h_\delta - M'J_\delta - (3H'' + 2I'' + 2K'')k_\delta + 2NJ'_\delta + 2N'k'_\delta,$$

(*l*) $$C_{333,\delta} = J_{,\delta} + 3(Ih_\delta - I'h'_\delta - I''J'_\delta), \tag{3.3}$$

(m) $$C_{334,\delta} = I'_{,\delta} + 2K'h_\delta + IJ_\delta + (J - 2J')h'_\delta - 2M''J'_\delta - I''k'_\delta,$$

(n) $$C_{335,\delta} = I''_{,\delta} + 2Nh_\delta - 2M''h'_\delta + (J - 2J'')J'_\delta + Ik_\delta + I'k'_\delta,$$

(o) $$C_{344,\delta} = J'_{,\delta} + Kh_\delta + 2K'J_\delta - (H - 2I')h'_\delta - K''J'_\delta - 2M''k'_\delta,$$

(p) $$C_{345,\delta} = M''_{,\delta} + N'h_\delta + NJ_\delta + (I'' - K'')h'_\delta + K'k_\delta + (I' - M')J'_\delta + (J' - J'')k'_\delta,$$

(q) $$C_{355,\delta} = J''_{,\delta} + Mh_\delta - M'h'_\delta + 2Nk_\delta - (H'' - 2I'')J'_\delta + 2M''k'_\delta,$$

(r) $$C_{444,\delta} = H'_{,\delta} + 3(KJ_\delta + J'h'_\delta - K''k'_\delta),$$

(s) $$C_{445,\delta} = K''_{,\delta} + 2N'J_\delta + 2M''h'_\delta + Kk_\delta + J'J'_\delta + (H' - 2M')k'_\delta,$$

(t) $$C_{455,\delta} = M_{,\delta} + MJ_\delta + J''h'_\delta + 2N'k_\delta + 2M'J'_\delta - (H'' - 2K'')k'_\delta,$$

(u) $$C_{555,\delta} = H''_{,\delta} + 3(Mk_\delta + J''J'_\delta + M'k'_\delta).$$

Adding (3.3)(c), (3.3)(*l*), (3.3)(o), (3.3)(q) and using (2.1), we get

$$C_{223,\delta} + C_{333,\delta} + C_{344,\delta} + C_{355,\delta} = LCh_\delta. \tag{3.4}$$

Adding (3.3)(d), (3.3)(m), (3.3)(r), (3.3)(t) and using (2.1), we get

$$C_{224,\delta} + C_{334,\delta} + C_{444,\delta} + C_{455,\delta} = LCJ_\delta. \tag{3.5}$$

Adding (3.3)(e), (3.3)(n), (3.3)(s), (3.3)(u) and using (2.1), we get

$$C_{225,\delta} + C_{335,\delta} + C_{445,\delta} + C_{555,\delta} = LCk_\delta. \tag{3.6}$$

Adding (3.3)(b), (3.3)(f), (3.3)(i), (3.3)(k) and using (2.1), we get

$$C_{222,\delta} + C_{233,\delta} + C_{244,\delta} + C_{255,\delta} = H_{,\delta} + I_{,\delta} + K_{,\delta} + M_{,\delta} = (H + I + K + M)_{,\delta} = (LC)_{,\delta}. \tag{3.7}$$

Definition. A Berwald space is characterized by $C_{hij|k} = 0$, which is given by in terms of scalar $C_{\alpha\beta\gamma,\delta} = 0$.

For a Berwald space, from (3.4), we have $h_\delta = 0$, from (3.5), we have $J_\delta = 0$ and from (3.6), we have $k_\delta = 0$ and so from (3.3b) $H_{,\delta} = 0$.

Theorem 3.1. A non-Riemannian Berwald space of dimension 5 is characterized by the fact that the three components h_i, J_i and k_i of the *h*–connection vector vanish and the main scalar H is $h-$ covariant constant.

Theorem 3.2. In a non-Riemannian Berwald space of dimension 5, the unified main scalar *LC* is $h-$ covariant constant.

Definition. A Landsberg space is characterized by

$$C_{hij|0} = 0 \quad \text{and} \quad C_{hij|k} = C_{hik|j}, \tag{3.8}$$

i.e., in terms of scalars $C_{\alpha\beta\gamma,1} = 0$ and $C_{\alpha\beta\gamma,\delta} = C_{\alpha\beta\delta,\gamma}$.

For a Landsberg space, from (3.4), (3.5) and (3.6), we get, $h_1 = 0$, $J_1 = 0$, $k_1 = 0$ and so $H_{,1} = 0$.

Also from (3.4), we have

$$\begin{aligned} LCh_2 &= C_{223,2} + C_{333,2} + C_{344,2} + C_{355,2} \\ &= C_{222,3} + C_{233,3} + C_{244,3} + C_{255,3} \\ &= (LC)_{,3} \qquad \text{from (3.7)} \end{aligned}$$

hence, $$h_2 = \frac{(LC)_{,3}}{LC}.$$

From (3.5), we have

$$\begin{aligned} LCJ_2 &= C_{422,2} + C_{433,2} + C_{444,2} + C_{455,2} \\ &= C_{222,4} + C_{233,4} + C_{244,4} + C_{255,4} \\ &= (LC)_{,4} \qquad \text{from (3.7)} \end{aligned}$$

hence, $$J_2 = \frac{(LC)_{,4}}{LC}.$$

Also from (3.5), we have

$$\begin{aligned} LCJ_3 &= C_{224,2} + C_{334,3} + C_{444,3} + C_{455,3} \\ &= C_{223,4} + C_{333,4} + C_{344,4} + C_{355,4} \\ &= (LC)h_4 \qquad \text{from (3.4)} \end{aligned}$$

hence, $J_3 = h_4$.

Also from (3.4), we have

$$\begin{aligned} LCh_5 &= C_{223,5} + C_{333,5} + C_{344,5} + C_{355,5} \\ &= C_{225,3} + C_{335,3} + C_{445,3} + C_{555,3} \\ &= (LC)k_3 \qquad \text{from (3.6)} \end{aligned}$$

hence, $h_5 = k_3$.

From (3.6), we have

$$\begin{aligned} LCk_2 &= C_{225,2} + C_{335,2} + C_{445,2} + C_{555,2} \\ &= C_{222,5} + C_{233,5} + C_{244,5} + C_{255,5} \\ &= (LC)_{,5} \qquad \text{from (3.7)} \end{aligned}$$

hence, $k_2 = \dfrac{(LC)_{,5}}{LC}$.

From (3.6), we have

$$\begin{aligned} LCk_4 &= C_{225,4} + C_{335,4} + C_{445,4} + C_{555,4} \\ &= C_{224,5} + C_{334,5} + C_{444,5} + C_{455,5} \\ &= LCJ_5 \qquad \text{from (3.5)} \end{aligned}$$

hence, $k_4 = J_5$.

Thus, we have

Theorem 3.3. In a non-Riemannian Landsberg space of dimension 5 the components of $h-$ connection vectors are given by

$$h_1 = J_1 = k_1 = 0, \qquad h_2 = \frac{(LC)_{,3}}{LC}, \qquad J_2 = \frac{(LC)_{,4}}{LC}, \qquad k_2 = \frac{(LC)_{,5}}{LC},$$
$$J_3 = h_4, \qquad k_4 = J_5, \qquad h_5 = k_3.$$

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