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Analysis of a Guided Deep Beam of Isotropic Materials Using a Mixed Boundary-Value Elasticity Approach

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Abstract : *An ideal mathematical model is developed to investigate guided simply-supported deep beam of isotropic material. The results are also analyzed with different aspect ratio on the distributions of displacement and stresses in the beam. Besides, taking into account the effect of Saint Venant's principle, a new analytical scheme is developed to generate the solution of a guided simply supported deep beam. Solutions of the guided isotropic deep beams are obtained satisfying all the physical conditions of the beam appropriately. Finally, comparative studies are carried out to ascertain the reliability of the present scheme solutions with those of classical beam theory and numerical method like the standard finite element method to verify the accuracy of the present modeling approach.*

Keywords : *Analytical model, displacement potential, simply supported beam, deep beam.*

I. INTRODUCTION

A beam is said to be a deep one when the beam thickness is comparable to its length. Design of deep beams based on classical Euler bending theory can be seriously erroneous, since the simple theory of flexure takes no account of the effect of normal pressures on the top and bottom edges of the beam caused by the loads and reactions. The effect of normal pressures on the stress distribution in these beams is such that the distribution of bending stresses on vertical sections is not linear and the distribution of shear stresses is not parabolic. Consequently, a plane transverse section does not remain plane after bending, and the neutral axis does not lie at the mid-depth, which eventually causes the basis of classical theory to be violated. In an attempt to make up the limitation, different theories as well as methods of solution have been reported in the literature [1-6]. However, each solution possesses certain limitations, and eventually none of the solutions are found to conform to all the physical conditions of the problem appropriately. Photoelastic [7], finite element analysis [8,9] as well as finite difference analysis [10,11] have been carried out for thick beams on two supports, mainly because all the physical conditions imposed on the beam could not be fully taken into account in the analytical methods of solution [9, 12].

Further, the use of standard structures, like beams, panels, columns, etc. with guides on part or full of their bounding surfaces is receiving increased importance in order to satisfy precise and strict design criteria in many of the engineering applications. The exact analytical solution of mixed-boundary-value elastic problems, especially with isotropic materials is beyond the scope of existing mathematical models of elasticity, the use of a new mathematical formulation will be investigated to analyze the elastic behavior of a guided deep beam of isotropic materials under standard loading and support arrangements. It would be worth mentioning that, as far as the reporting in the literature is concerned, the author has not come across even any reliable study of the present problem, either theoretical or an experimental one. Results of displacements and stresses are obtained for beams with uniformly distributed loading and are analyzed in the perspectives of beam depth, length and aspect ratio. The reliability and appropriateness of the present solutions are discussed in light of comparisons made with available theoretical solutions in the literature, which includes the classical bending theory and finite-element method.

II. MATHEMATICAL BACKGROUND

Analysis of stress in a material body is generally a 3-D problem. But in the most of the cases, these 3-D bodies can easily be treated as 2-D problem since most practical problems are found to conform to the states of plane stress or plain strain. In case of the absence of any body forces, the governing equations of the three stress components σ_{xx}, σ_{yy} and σ_{xy} under the states of plane stress or plain strain are:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 \tag{1a}$$

$$\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} = 0 \tag{1b}$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\sigma_{xx}+\sigma_{yy})=0 \tag{1c}$$

If the stress functions be replaced in equations (1a) to (1c) by displacement function u_x and u_y, which are related to stress functions through the expressions,

$$\sigma_{xx}=\frac{E}{1-\mu^2}\left[\frac{\partial u_x}{\partial x}+\mu\frac{\partial u_y}{\partial y}\right] \tag{2a}$$

$$\sigma_{yy}=\frac{E}{1-\mu^2}\left[\mu\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}\right] \tag{2b}$$

$$\sigma_{xy}=\frac{E}{2(1+\mu)}\left[\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right] \tag{2c}$$

then equation (1c) is redundant and equations (1a) and (1b) transform to:

$$\frac{\partial^2 u_x}{\partial x^2}+\left(\frac{1-\mu}{2}\right)\frac{\partial^2 u_x}{\partial y^2}+\left(\frac{1+\mu}{2}\right)\frac{\partial^2 u_y}{\partial x\partial y}=0 \tag{3a}$$

$$\frac{\partial^2 u_y}{\partial y^2}+\left(\frac{1-\mu}{2}\right)\frac{\partial^2 u_y}{\partial x^2}+\left(\frac{1+\mu}{2}\right)\frac{\partial^2 u_x}{\partial x\partial y}=0 \tag{3b}$$

The problem thus reduces to finding u_x and u_y in a 2-D field satisfying the two elliptic partial differential equations, (3a) and (3b).

The problem is reduced to the determination of a single function instead of two functions u_x and u_y simultaneously, satisfying the equilibrium equations (3a) and (3b). In this formulation, as in the case of Airy's stress function ϕ a potential function $\psi(x,y)$ is defined in terms of displacement components as

$$u_x=\frac{\partial^2\psi}{\partial x\partial y}$$

$$u_y=\frac{1}{1+\mu}\left[(1-\mu)\frac{\partial^2\psi}{\partial y^2}+2\frac{\partial^2\psi}{\partial x^2}\right]$$

When the displacement components in the equations (3a) and (3b) are replaced by $\psi(x,y)$, equation (3a) is automatically satisfied and the only condition that $\psi(x,y)$ has to satisfy, becomes

$$\frac{\partial^4\psi}{\partial x^4}+2\frac{\partial^4\psi}{\partial x^2\partial y^2}+\frac{\partial^4\psi}{\partial y^4}=0 \tag{4}$$

Since the present approach considers a single dependent variable for the plane beam problem, all the equations associated with the normal and the tangential components of stress and displacement are expressed in terms of the function ψ.

The components of different displacement components are as follows:

$$u_x(x,y)=\frac{\partial^2\psi}{\partial x\partial y} \tag{5}$$

$$u_y(x,y)=-\frac{1}{1+\mu}\left[2\frac{\partial^2\psi}{\partial x^2}+(1-\mu)\frac{\partial^2\psi}{\partial y^2}\right] \tag{6}$$

The expressions of stress components in terms of the function ψ are as follows:

$$\sigma_{xx}(x,y)=\frac{E}{(1+\mu)^2}\left[\frac{\partial^3\psi}{\partial x^2\partial y}-\mu\frac{\partial^3\psi}{\partial y^3}\right] \tag{7}$$

$$\sigma_{yy}(x,y) = -\frac{E}{(1+\mu)^2}\left[(2+\mu)\frac{\partial^3\psi}{\partial x^2\partial y} + \frac{\partial^3\psi}{\partial y^3}\right] \tag{8}$$

$$\sigma_{xy}(x,y) = \frac{E}{(1+\mu)^2}\left[-\frac{\partial^3\psi}{\partial x^3} + \mu\frac{\partial^3\psi}{\partial x\partial y^2}\right] \tag{9}$$

It is mentioned that this partial differential equation has been extensively used for the solution of mixed-boundary-value elastic problems of isotropic materials.

III. DESCRIPTION OF THE BEAM PROBLEM

A simply supported deep beam of rectangular cross section subjected to a distributed load is considered. The generalized form of such a beam is shown in Fig. 1(a). While the two opposing lateral ends of the beam are guided by frictionless rollers illustrates in Fig 1(b). The support at the bottom surface is located at the two corner regions of the beam. The length, depth and width of the beam are denoted by *L*, *D* and *W*, respectively. The load acting over the top surface is considered as uniformly distributed with a magnitude of σ_0 acting over the length of 0.8*L*. The support of the beam is also considered as uniformly distributed and the effective length for each support is assumed to be 0.1*L*. The plane stress is assumed here taking unit thickness of the beam. Since the two opposing lateral ends of the beam are guided, the axial displacements are restrained, but the lateral displacements are free to assume any value.

IV. BOUNDARY CONDITIONS

The physical conditions of the present problem with reference to Fig. 2 are to be satisfied along the all four boundaries of the beam. The boundary conditions are considered for the present stiffened composite beam problem can be expressed mathematically as follows:

(a) Loaded boundary, EH:

The loading of the top boundary is modelled by assigning a uniform value to the normal stress component, which is free from any shearing stress. The mathematical expressions of the boundary are

$$\sigma_{yy}(x,D) = \sigma_0 \quad [0.1 \le x/L \le 0.9]$$

$$\sigma_{xy}(x,D) = 0 \quad [0.0 \le x/L \le 1.0]$$

(b) Supporting surface, FG:

The roller supported regions of the bottom surface are modelled by a uniform compressive normal loading and free from shearing stress. At the supports, the total reaction forces should be equal and opposite to the applied loading on the top surface. The reactions are distributed over 20% of the beam span (*x/L*= 0.0~0.1 and 0.9~1.0).The remaining section of the bottom surface are assumed to be free from loading. Therefore,

Supporting region: $\sigma_{xy}(x,0) = 0 \quad [0.0 \le x/L \le 1.0]$

$$\sigma_{yy}(x,0) = 4\sigma_0 \quad [0.0 \le x/L \le 0.1 \,\&\, 0.9 \le x/L \le 1.0]$$

Free region: $\sigma_{xy}(x,0) = 0 \quad [0.0 \le x/L \le 1.0]$

$$\sigma_{yy}(x,0) = 0 \quad [0.1 < x/L < 0.9]$$

(c) Left lateral end, EF:

The physical condition of the roller guide is modeled here by considering no axial displacement and shearing stress. Thus,

$$u_x(0,y) = 0$$

$$\sigma_{xy}(0,y) = 0 \; [0.0 \le y/D \le 1.0]$$

(d) Left lateral end, EF:

The mathematical expressions of this boundary are

$$u_x(L,y) = 0$$

$$\sigma_{xy}(L,y) = 0 \quad [0.0 \le y/D \le 1.0]$$

V. ANALYTICAL SOLUTION

The potential function $\psi(x, y)$ is first assumed in a way so that the physical conditions of the two opposing guided ends are automatically satisfied. The solution of the governing equation (4) is thus approximated as

$$\psi(x, y) = \sum_{m=1}^{\infty} Y_m(y)\cos\alpha x + K y^3 \tag{10}$$

where, $Y_m = f(y)$, $\alpha = (m\pi / L)$, K is an arbitrary constant and m= 1, 2, 3, ……..∞

Now, substitution of Eq. (10) into Eq. (4) yields the following ordinary differential equation in Y_m,

$$\frac{d^4Y_m}{dx^4} - 2\alpha^2\frac{d^2Y_m}{dx^2} + \alpha^2 Y_m = 0 \tag{11}$$

The above differential equation with constant coefficients has the complementary function of two repeated roots (r). Thus, $r_1 = r_2 = \alpha$ and $r_3 = r_4 = -\alpha$, the general solution of Eq. (11) can be approximated as follows:

$$Y_m = (A_m + B_m y)e^{\alpha y} + (C_m + D_m y)e^{-\alpha y} \tag{12}$$

where A_m, B_m, C_m and D_m are arbitrary constants.

Now substituting the derivatives of ψ and Y_m in the expressions for displacement (5~6) and stresses (7~9), following expressions are found.

$$u_x(x, y) = -\sum_{m=1}^{\infty}\left[A_m\alpha e^{\alpha y} + B_m(\alpha y + 1)e^{\alpha y} - C_m\alpha e^{-\alpha y} - D_m(\alpha y - 1)e^{-\alpha y}\right]\alpha \sin \alpha x \tag{13}$$

$$u_y(x, y) = \frac{-1}{(1+\mu)}\left[\sum_{m=1}^{\infty}\left\{\begin{aligned} &-A_m(1+\mu)\alpha^2 e^{\alpha y} + \\ &B_m(-\alpha y - \mu\alpha y - 2\mu + 2)\alpha e^{\alpha y} \\ &-C_m(1+\mu)\alpha^2 e^{-\alpha y} + \\ &D_m(-\alpha y - \mu\alpha y + 2\mu - 2)\alpha e^{-\alpha y}\end{aligned}\right\}\cos\alpha x + 6K(1-\mu)y\right]\sin\alpha x \tag{14}$$

$$\sigma_{xx}(x, y) = \frac{-E}{(1+\mu)^2}\left[\sum_{m=1}^{\infty}\left\{\begin{aligned} &A_m\alpha(1+\mu)e^{\alpha y} + B_m(\alpha y + \mu\alpha y + 3\mu + 1)e^{\alpha y} \\ &-C_m\alpha(1+\mu)e^{-\alpha y} + D_m(-\alpha y - \mu\alpha y + 3\mu + 1)e^{-\alpha y}\end{aligned}\right\}\alpha^2\cos\alpha x + 6\mu K\right] \tag{15}$$

$$\sigma_{yy}(x, y) = \frac{-E}{(1+\mu)^2}\left[\sum_{m=1}^{\infty}\left\{\begin{aligned} &A_m\alpha(-1-\mu)e^{\alpha y} + B_m(-\alpha y - \mu\alpha y - \mu + 1)e^{\alpha y} + \\ &C_m\alpha(1+\mu)e^{-\alpha y} + D_m(\alpha y + \mu\alpha y - \mu + 1)e^{-\alpha y}\end{aligned}\right\}\alpha^2\cos\alpha x + 6K\right] \tag{16}$$

$$\sigma_{xy}(x, y) = \frac{-E}{(1+\mu)^2}\left[\sum_{m=1}^{\infty}\left\{\begin{aligned} &A_m(1+\mu)\alpha e^{\alpha y} + B_m(\alpha y + \mu\alpha y + 2\mu)e^{\alpha y} \\ &+C_m(1+\mu)\alpha e^{-\alpha y} + D_m(\alpha y + \mu\alpha y - 2\mu)e^{-\alpha y}\end{aligned}\right\}\alpha^2\sin\alpha x\right] \tag{17}$$

Now, the reactions on the bottom boundary (y = 0) are acting over the two supports. It is considered that the supports are located at x=0 to 0.1L and x=0.9L to L respectively. The total length for reaction is 20% of beam length, where the load is over the 80%. As a result the intensity of reaction is four times of the load intensity. Therefore, the reactions over the beam at the supports can be taken as Fourier series in the following manner:

$$\sigma_{yy}(x,0) = 4\sigma_0 = E_0 + \sum_{m=1}^{\infty} E_m\cos\alpha x \quad \text{for } x = 0 \text{ to } 0.1L \text{ and } 0.9L \text{ to } L \tag{18}$$

Here

$$E_0 = \frac{1}{L}\left[\int_0^{L/10} 4\sigma_0\,dx + \int_{9L/10}^{L} 4\sigma_0\,dx\right]$$

$$= \frac{4\sigma_0}{5}$$

$$E_m = \frac{2}{L}\left[\int_0^{L/10} 4\sigma_0 \cos\alpha x dx + \int_{9L/10}^{L} 4\sigma_0 \cos\alpha x dx\right]$$

$$= \frac{8\sigma_0}{m\pi}\left\{\sin\left(\frac{m\pi}{10}\right) + \sin(m\pi) - \sin\left(\frac{9m\pi}{10}\right)\right\}; \qquad m = 1,2,3, \ldots. \infty$$

The compressive load on the edge $y = D$ acting over $x = 0.1L$ to $0.9L$ can also be given by a Fourier series as follows

$$\sigma_{yy}(x,0) = \sigma_0 = I_0 + \sum_{m=1}^{\infty} I_m \cos\alpha x \quad \text{for } x = 0.1L \text{ to } 0.9L \tag{19}$$

Here

$$I_0 = \frac{1}{L}\left[\int_{L/10}^{9L/10} \sigma_0 dx\right]$$

$$= \frac{4\sigma_0}{5}$$

$$I_m = \frac{2}{L}\left[\int_{L/10}^{9L/10} \sigma_0 \cos\alpha x dx\right]$$

$$= \frac{2\sigma_0}{m\pi}\left\{\sin\left(\frac{9m\pi}{10}\right) - \sin\left(\frac{m\pi}{10}\right)\right\}; \qquad m = 1,2,3, \ldots\ldots\ldots \infty$$

The loading considerations of equations (18) and (19) are to satisfy the boundary conditions at the bottom and top boundaries of the beam. Using boundary condition $s_{xy}(x,0) = 0$ at the edge of $y = 0$, it is found that

$$\frac{-E\alpha^2}{(1+\mu)^2}\left[(1+\mu)\alpha A_m + 2\mu B_m + (1+\mu)\alpha C_m - 2\mu D_m\right] = 0 \tag{20a}$$

Using boundary condition $\sigma_{xy}(x, D) = 0$ at the edge of $y = D$,

$$\frac{-E\alpha^2}{(1+\mu)^2}\left[\begin{matrix} A_m(1+\mu)\alpha e^{\alpha D} + B_m(\alpha D + \mu\alpha D + 2\mu)e^{\alpha D} + \\ C_m(1+\mu)\alpha e^{-\alpha D} + D_m(\alpha D + \mu\alpha D - 2\mu)e^{-\alpha D} \end{matrix}\right] = 0 \tag{20b}$$

Using boundary condition $\sigma_{yy}(x,0) = 4\sigma_0$ at the edge of $y = 0$,

$$\frac{E\alpha^2}{(1+\mu)^2}\left[A_m(1+\mu)\alpha + B_m(-1+\mu) - C_m(1+\mu)\alpha + D_m(-1+\mu)\right] = E_m \tag{20c}$$

Using boundary condition $\sigma_{yy}(x, D) = \sigma_0$ at the edge of $y = D$

$$\frac{E\alpha^2}{(1+\mu)^2}\left[\begin{matrix} A_m(1+\mu)\alpha e^{\alpha D} + B_m(\mu\alpha D + \alpha D + \mu - 1)e^{\alpha D} - \\ C_m(1+\mu)\alpha e^{-\alpha D} - D_m(\mu\alpha D + \alpha D - \mu + 1)e^{-\alpha D} \end{matrix}\right] = I_m \tag{20d}$$

The arbitrary constant *K* can be obtained as follows:

$$\frac{-E}{(1+\mu)^2}6K = E_0 = \frac{4\sigma_0}{5}$$

$$\text{or, } K = \frac{-2\sigma_0(1+\mu)^2}{15E}$$

The simultaneous solution of the above four algebraic equations (20a) ~ (20b) yields the four arbitrary constants assumed in the solution. Once the values of the unknown constants, that is, A_m, B_m, C_m and D_m, are known, the solution of the elastic field, that is, expressions of displacement and stress components are explicitly known as a function of the coordinate parameters, *x* and *y*.

VI. COMPARATIVE ANALYSIS OF SOLUTIONS

The analytical solutions of displacement and stress components are obtained for various aspect ratios (*L/D*) of the beam. The result of a guided steel beam (μ = 0.3 and *E* = 209 GPa) and the uniform loading parameter, σ_0 = 40 N/mm. Then a comparative study is presented to validate of the displacement potential solutions over the classical beam theory as well as finite element method (FEM). Finally the effects of the beam aspect ratio on the elastic fields are analyzed.

i) Classical Beam Theory:

The classical theory of banding (simple beam theory) basically gives a one-dimensional solution to the beam problem, which includes the lateral deflection, bending stress and shearing stress as a function of beam length and depth. The solutions so obtained as a function of beam aspect ratio are presented in a normalized form as follows:

$$\frac{u_y(x)}{D} = \frac{\sigma_0}{2E}\left(\frac{L}{D}\right)^4\left[2\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right)^3 - 1\right]\left(\frac{x}{L}\right) \tag{21}$$

$$\frac{\sigma_{xx}(x)}{\sigma_0} = \frac{24}{5}\left(\frac{L}{D}\right)^2\left(\frac{x}{L} - \frac{1}{20}\right)\left(\frac{1}{2} - \frac{y}{D}\right) \qquad [0 \le x/L < 0.1 \text{ and } 0.9 < x/L \le 1.0] \tag{22}$$

$$= 6\left(\frac{L}{D}\right)^2\left[\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right) + \frac{1}{25}\right]\left(\frac{1}{2} - \frac{y}{D}\right) \qquad [0.1 \le x/L \le 0.9] \tag{23}$$

$$\frac{\sigma_{xy}(x)}{\sigma_0} = 6\left(\frac{L}{D}\right)\left[\frac{1}{4} - \left(\frac{y}{D} - \frac{1}{2}\right)^2\right]\left(\frac{1}{2} - \frac{x}{L}\right) \qquad [0.0 \le x/L \le 1.0] \tag{24}$$

ii) Finite Element Method

Finite element method is widely used all over the world for various computational purposes in lab and commercial areas. In this study, ANSYS has been used to solve several problems in order to compare and verify the analytical results. The relevant boundary conditions used are the same as those used in the analytical solution. Four nodded rectangular plane elements are used to construct the corresponding mesh network of the beam. The total number of finite elements used to construct the element mesh network for all problems is 6400 (80x80).

The comparison of the present ψ - solution with those obtained by classical beam theory and FEM result is presented in Fig. 2. Fig. 2 (a) illustrates that the bending stress distribution changed linearly for beam theory and overestimated from those of the corresponding ψ - solution as well as FEM solution. Fig. 2 (b) presents the shear stress distribution at section of *x/L* = 0.25 obtained by the three approaches is found very close to each other. As appears from Fig. 2 (c), the deflection is underestimated by the classical beam theory, whereas that of FEM is very close to the present solution.

Fig. 3 presents the deformed shape of the beam together with the original shape with the magnification of 500 times of displacement. The stiffened ends have gone up and at the same time centre region of the beam have gone down. The deformation of the top edge is uniform throughout the length of beam with the uniformly distributed loading. The bottom edge is also deformed uniformly except the support region, where there is very little non-uniformity of deformation. However, the overall vertical sliding type deformation is again in excellent agreement with the applied loading and support of the beam.

VII. FIGURES

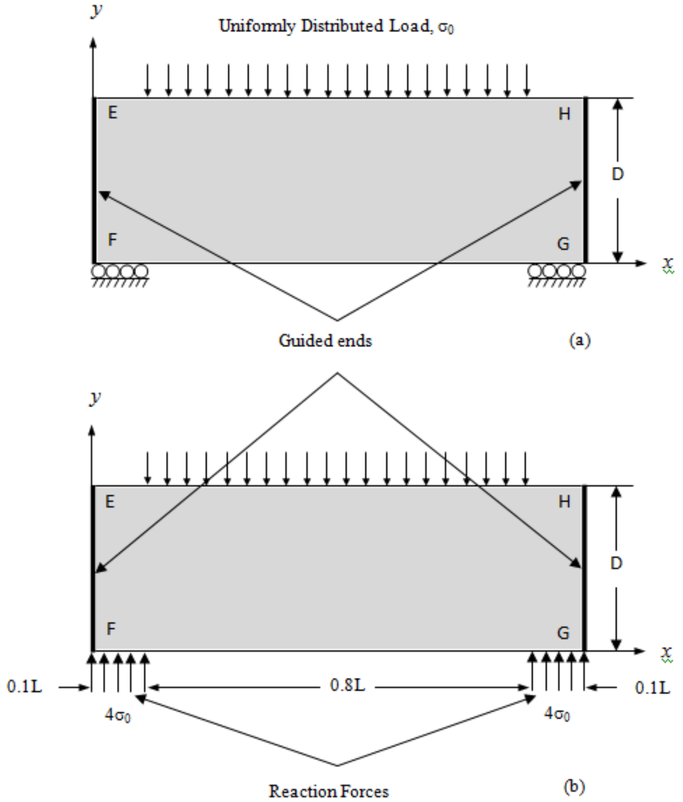
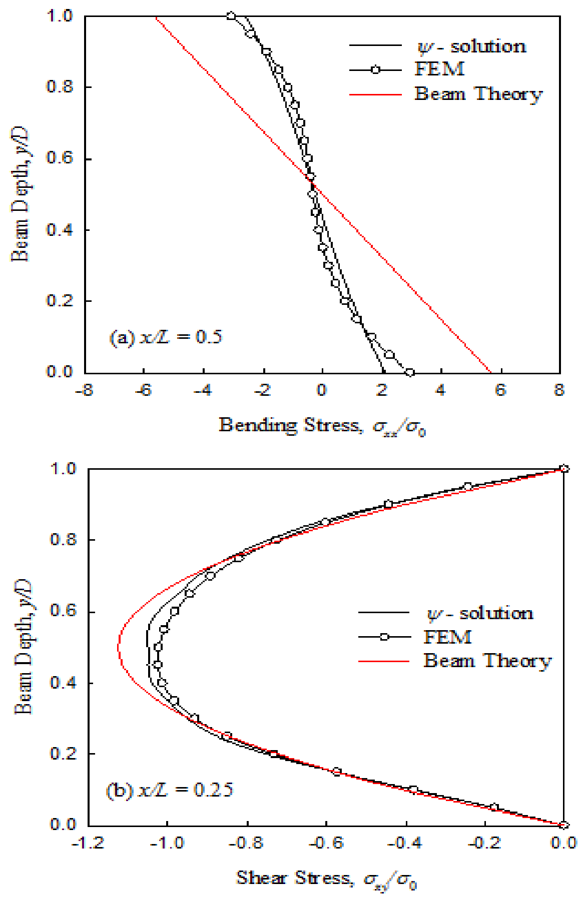


Fig. 1 (a) Physical model and (b) Analytical model of a stiffened simply-supported beam of isotropic material



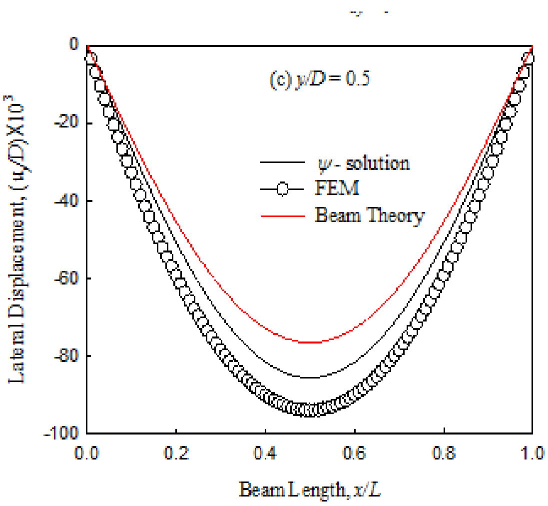


Fig. 2 Comparison of solutions for (a) bending stress, (b) shear stress and (c) lateral displacement of the isotropic beam for L/D = 3

VIII. CONCLUSION

The central objective of this paper is to develop the analytical solutions for guided isotropic simply supported deep beams. Here an attempt is made to remove the limitation of the literature by developing a new analytical scheme of simply supported isotropic beam. In order to check the reliability and accuracy of the present analytical solutions, the results are compared with the numerical and classical beam theory. It is observed that the solutions are in excellent agreement with each other.

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