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Operator's Differential geometry with Riemannian Manifolds

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ABSTRACT : In this paper some fundamental theorems , operators differential geometry – with operator Riemannian geometry to pervious of differentiable manifolds which are used in an essential way in basic concepts of Spectrum of Discrete , bounded Riemannian geometry, we study the defections, examples of the problem of differentially projection mapping parameterization system on dimensional manifolds .

Keywords *-Basic notions on differential geometry-The spectral geometry –The geometric global minima and maxima-The geometric of Laplace and Dirac spinner Bounded – Heat trace Asymptotic closed manifolds – Riemannian manifolds with same spectrum Bounded Harmonic function – compact Riemannian manifolds – computations of spectrum.*

I. Introduction

Differential forms and the exterior derivative provide one piece of analysis on manifolds which, as we have seen , links in with global topological questions . There is much more on can do when on introduces a Riemannian metric. Since the whole subject of Riemannian geometry is a huge to the use of differential forms. The study of harmonic from and of geodesics in particular, we ignore completely hare questions related to curvature. The spectrum does not in general determine the geometry of a manifold Nevertheless earthiness, some geometric information can be extracted from the spectrum. In what follows, we define a spectral invariant to be anything that is completely determined by the spectrum .A Riemannian manifold is a pair $(M .g)$ consisting of a smooth manifold M and a metric g on the tangent bundle, i.e. a smooth symmetric positive definite tensor field on M . The tensor g is called a Riemannian metric on M

II. Basic Notions On Differential Geometry

2.1 Basic on topological Manifold

Definition 2.1.1 Topological Manifold

A topological manifold M of dimension n , is a topological space with the following properties:

(a) M Is a Hausdorff space . For ever pair of points $p, g \in M$, there are disjoint open subsets $U, V \subset M$ such that $p \in U$ and $g \in V$.

(b) M Is second countable. There exists accountable basis for the topology of M . (c) M is locally Euclidean of dimension n . Every point of M has a neighborhood that is homeomorphism to an open subset of R^n .

Definition 2.1.3

A topological space M is called an m-dimensional topological manifold with boundary $\partial M \subset M$ if the following conditions : (i) M is Hausdorff space.(ii) for any point $p \in M$ there exists a neighborhood U of p which is homeomorphism to an open subset $V \subset H^m$.(iii) M has a countable basis of open sets , can be rephrased as follows any point $p \in U$ is contained in neighborhood U to $D^m \cap H^m$ the set M is a locally homeomorphism to R^m or H^m the boundary $\partial M \subset M$ is subset of M which consists of points p .

Definition 2.1.4

Let X be a set a topology U for X is collection of X satisfying : (i) ϕ and X are in U (ii) the intersection of two members of U is in U .(iii) the union of any number of members U is in U . The set X with U is called a topological space the members $U \in u$ are called the open sets . let X be a topological space a subset $N \subseteq X$ with $x \in N$ is called a neighborhood of x if there is an open set U with $x \in U \subseteq N$, for example if X a metric space then the closed ball $D_\varepsilon(x)$ and the open ball $D_\varepsilon(x)$ are neighborhoods of x a subset C is said to closed if $X \setminus C$ is open

Definition 2.1.5

A function $f : X \rightarrow Y$ between two topological spaces is said to be continuous if for every open set U of Y the pre-image $f^{-1}(U)$ is open in X .

Definition 2.1.6

Let X and Y be topological spaces we say that X and Y are homeomorphic if there exist continuous function such that $f \circ g = id_y$ and $g \circ f = id_x$ we write $X \cong Y$ and say that f and g are homeomorphisms between X and Y, by the definition a function $f : X \to Y$ is a homeomorphisms if and only if .(i) f is a bijective .(ii) f is continuous (iii) f^{-1} is also continuous.

Definition 2.1.7 Coordinate Charts

A coordinate chart or just a chart on a topological $n-$ manifold M is a pair (U, φ), Where U is an open subset of M and $\varphi : U \to \tilde{U}$ is a homeomorphism from U to an open subset $\tilde{U} = \varphi(U) \subset R^n$.

Examples 2.1.8 Topological Manifolds] Spheres

Let S^n denote the (unit) $n-$ sphere, which is the set of unit vectors in R^{n+1}: $S^n = \{x \in R^{n+1} : |x| = 1\}$ with the subspace topology, S^n is a topological $n-$ manifold.

Definition 2.1.9 Projective spaces

The $n-$ dimensional real (complex) projective space, denoted by $P_n(R)$ or $P_n(C)$), is defined as the set of 1-dimensional linear subspace of R^{n+1} or C^{n+1}), $P_n(R)$ or $P_n(C)$ is a topological manifold.

Definition 2.1.10

For any positive integer n, the $n-$ torus is the product space $T^n = (S^1 \times \ldots \times S^1)$ .It is an $n-$ dimensional topological manifold. (The 2-torus is usually called simply the torus).

Definition2.1.11 Boundary of a manifold

The boundary of a line segment is the two end points; the boundary of a disc is a circle. In general the boundary of an $n-$ manifold is a manifold of dimension $(n-1)$, we denote the boundary of a manifold M as ∂M . The boundary of boundary is always empty, $\partial\partial M = \phi$

Lemma 2.1.12

(i)Every topological manifold has a countable basis of Compact coordinate balls. (ii) Every topological manifold is locally compact.

Definitions 2.1.13 [Transition Map]

Let M be a topological space n -manifold. If $(U, \varphi), (V, \psi)$ are two charts such that $U \cap V \neq \phi$, the composite map.

(1) $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$

Is called the transition map from φ to ψ .

Definition 2.1.14 [A smooth Atlas]

An atlas A is called a smooth atlas if any two charts in A are smoothly compatible with each other. A smooth atlas A on a topological manifold M is maximal if it is not contained in any strictly larger smooth atlas. (This just means that any chart that is smoothly compatible with every chart in A is already in A.

Definition 2.1.15 [A smooth Structure]

A smooth structure on a topological manifold M is maximal smooth atlas. (Smooth structure are also called differentiable structure or C^∞ structure by some authors).

Definition 2.1.16 [A smooth Manifold]

A smooth manifold is a pair $(M$, A), where M is a topological manifold and A is smooth structure on M . When the smooth structure is understood, we omit mention of it and just say M is a smooth manifold.

Definition 2.1.17

Let M be a topological manifold: (i)Every smooth atlases for M is contained in a unique maximal smooth atlas. (ii) Two smooth atlases for M determine the same maximal smooth atlas if and only if their union is smooth atlas.

Definition 2.1.18

Every smooth manifold has a countable basis of pre-compact smooth coordinate balls. For example the General Linear Group The general linear group $GL(n, R)$ is the set of invertible $n \times n$ -matrices with real entries. It is a smooth n^2 -dimensional manifold because it is an open subset of the n^2 - dimensional vector space $M(n, R)$, namely the set where the (continuous) determinant function is nonzero.

Definition 2.1.19[Tangent Vectors on A manifold]

Let M be a smooth manifold and let p be a point of M A linear map $X : C^\infty(M) \to R$ is called a derivation at p if it satisfies:

(2) $X(fg) = f(p)Xg + g(p)Xf$

Forall $f, g \in C^\infty(M)$. The set of all derivation of $C^\infty(M)$ at p is vector space called the tangent space to M at p, and is denoted by [T_pM]. An element of T_pM is called a tangent vector at p .

Lemma 2.1.20 [Properties of Tangent Vectors]
Let M be a smooth manifold, and suppose $p \in M$ and $X \in T_p M$. If f is a const and function, then $Xf = 0$. If

(3) $f(p) = g(p) = 0$, then $X(fp) = 0$.

Definition2.1.22 [Tangent Vectors to Smooth Curves]
If γ is a smooth curve (a continuous map $\gamma : J \to M$,where $J \subset R$ is an interval) in a smooth manifold M , we define the tangent vector to γ at $t_\circ \in J$ to be the vector .

(4) $$\gamma'(t_\circ) = \gamma_* \left(\frac{d}{dt} |_{t_\circ} \right) \in T_{\gamma(t_\circ)} M$$

where $d/dt\ |_{t_\circ}$ is the standard coordinate basis for $T_{t_\circ} R$. Other common notations for the tangent vector to γ are $\left[\gamma^{\cdot}(t_\circ), \frac{d\gamma}{dt}(t_\circ) \right]$ and $\left[\frac{d\gamma}{dt} |_{t = t_\circ} \right]$. This tangent vector acts on functions by :

(5) $$\gamma'(t_\circ)\ f = \left(\gamma_* \frac{d}{dt} |_{t_\circ} \right) f = \frac{d}{dt} |_{t_\circ} (f \circ \gamma) = \frac{d(f \circ \gamma)}{dt}(t_\circ)$$

Lemma 2.1.23
Let M be a smooth manifold and $p \in M$ .Every $X \in (T_p M)$ is the tangent vector to some smooth curve in M .

Definition 2.1.24 [Lie Groups]
A Lie group is a smooth manifold G that is also a group in the algebraic sense, with the property that the multiplication map $m : G \times G \to G$ and inversion map $m : G \to G$, given by $m(g,h) = gh,\ \ i(g) = g^{-1}$, are both smooth. If G is a smooth manifold with group structure such that the map $G \times G \to G$ given by $(g,h) \to gh^{-1}$ is smooth, then G is a Lie group. Each of the following manifolds is a lie group with indicated group operation. The general linear group $GL(n,R)$ is the set of invertible $n \times n$ matrices with real entries. It is a group under matrix multiplication, and it is an open sub-manifold of the vector space $M(n,R)$, multiplication is smooth because the matrix entries of A and B . Inversion is smooth because Cramer's rule expresses the entries of A^{-1} as rational functions of the entries of A . The $n-$ torus $T^n = (S^1 \times ... \times S^1)$ is an $n-$ dimensional a Belgian group.

Definition 2.1.25 [Generalized Tensor is Riemannian]
If an m-dimensional smooth manifold M is given a smooth every no degenerate symmetric covariant tensor field of rank-2 , G then M is called a generalized tensor or metric tensor or metric of M . If G is positive definite then M is called Riemannian manifold for a generalized Riemannian manifold $M\ , G = g_{ij}\, du^i \otimes du^j$ specifies an inner product on the tangent space $T_p(M)$ at every point $p \in M$ for any $X, Y \in T_p(M)$.

(6) $$X.Y = G(X.Y) = g_{ij}(p) X^i Y^j$$

When G is positive definite, it is meaningful to define the length of a tangent vector and the angle between two tangent vectors at the some point $|X| = \sqrt{g_{ij} X^i X^J}$. Thus a Riemannian manifold is a differentiable manifold which has a positive definite inner product on the tangent space at every point. The inner product is required to smooth X, Y are smooth tangent vector fields then X, Y is a smooth on M

Definition 2.1.26 [Smooth Parameterize Curve]
$dS^2 = g_{ij} du^i du^j$ is independent of the choice of the local coordinate system u^i and usually called the metric form or Riemannian metric (dS) is precisely the length of an infinitesimal tangent vector and is called the element of are length . Suppose a $C = u^i = u^i(t)$ and $t_0 \le t \le t_1$ is a continuous and piecewise smooth parameterized curve on M ,then the are length of C is defined to be .

(7) $$\left[S = \int_{t_0}^{t_1} \sqrt{g_{ij} \frac{du^i}{dt} \frac{du^j}{dt}} dt \right]$$

Remark 2.1.27
Exist a smooth is nonzero everywhere. The existence of a Riemannian metric on a smooth manifold is an extraordinary result. In general there may not exist a non-positive. In the context of fiber bundles , the existence of a Riemannian metric on M implies the existence of a positive definite smooth of bundle of symmetric covariant tensor of order 2-on M, However for arbitrary vector bundles there may not exist a smooth which is nonzero everywhere.

Theorem 2.1.28
Suppose M is an m-dimensional generalized Riemannian manifold then there exists a unique tensor – Free and metric compatible connection on M , called the (Levi-civet connecting of M) Riemannian connection of M
Proof:

Suppose D is a torsion-free and metric – compatible connection on M , denote the connection matrix of D under the local coordinates U^i by $W = (W_i^{\ j})$ where $W_i^{\ j} = \Gamma_{ik}^j du^k$ .Then we have $dg_{ij} = W_i^{\ j} g_{kj} + W_j^{\ k} g_{ki}$, and $\Gamma_{ik}^j = \Gamma_{kj}^j$ Denote that $\Gamma_{ik}^j = g_{ij}\Gamma_{kj}^j$, $W_{ik} = g_{lk}W_i^{\ l}$. Then its follows from that.

(8) $$\frac{\partial g_{ij}}{\partial u^k} = \left(\Gamma_{jik} + \Gamma_{ikj}^j\right)$$

$\Gamma_{jik} = \Gamma_{ikj}^j$ is cycling the indices in we get $\frac{\partial g_{ik}}{\partial u^j} = \Gamma_{jik} + \Gamma_{ikj}^j$ and $\frac{\partial g_{ik}}{\partial u^i} = \Gamma_{jik} + \Gamma_{ikj}^j$ .And calculating we then obtain .

(9) $$\left\{ \Gamma_{ij} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial u_j} + \frac{\partial g_{jk}}{\partial u_i} - \frac{\partial g_{jk}}{\partial u_k}\right) \quad \Gamma^k{}_{ji} = \frac{1}{2} g^{kl}\left(\frac{\partial g_{il}}{\partial u^j} + \frac{\partial g_{jl}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l}\right)\right\}$$

The equation is (Levi-civet connecting of M) or (Riemannian connection of M)

Definition 2.1.29 [Smooth Curve in M]

Let M be a Riemannian manifold and $\gamma : [0,1] \to M$ a smooth map i,e a smooth curve in M . The length of curve is $L(\gamma)$ and $F(Z) = \left(\frac{az + b}{cz + d}\right)$ With a, b, c, d and $ad - bc \geq 0$, then $Fdz = (ad - bc)\frac{dz}{(cz + d)^2}$ and .

(10) $$\begin{cases} Fg = (ad - bc)^2 \dfrac{dx + dy}{\left|(cz + d)^2\right|^2} \cdot \dfrac{\left|cz + d\right|^2}{(ad - bc)^2 y^2} \\ = \dfrac{dx^2 + dy^2}{y^2} = g \end{cases}$$

So these Movies transformation are isometrics of Riemannian metric on the upper half-plan.

2.3 : The Spectral Geometry of operators of Dirac and Laplace Type

We have also given in each a few additional references to relevant. The constraints of space have of necessity forced us to omit many more important references that it was possible to include and we a apologize in a dance for that. We a the following notational conventions , let (M, g) (be compact Riemannian manifold of dim. M with boundary ∂M .Let Greek indices γ, μ range from 1 *to* m , and index a local system of coordinates $x = (x^1, \ldots\ldots, x^m)$ on the interior of M expand the metric in the form $dS^2 = g_{\mu\gamma} dx^\mu dx^\gamma$ were $g_{\mu\gamma} = \left(\partial_{x_\mu}, \partial_{x_\gamma}\right)$ and where we adopt the Einstein convention of summing over repeated indices we let $g^{\mu\gamma}$ be the inverse matrix the Riemannian measure is given by $dx = (dx^1, \ldots\ldots, dx^m)$ for $g = \sqrt{\det(g_{\mu\gamma})}$ let ∇ be the"levi-Civita" connection. We expand $\nabla_{\partial x,\gamma} . \partial_{x\mu} = \Gamma_{\gamma\mu}\sigma\, \partial x_\sigma$. Where $\Gamma_{\gamma\mu}\sigma$ are the m, R are may then be given by. $R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$

And given by.

(11) $$R(X,Y,Z,W) = g(R(X,Y), Z, W)$$

We shall let Latin indices i, j range from 1 to m and index a local orthonormal frame $\{e_1, \ldots\ldots, e_m\}$ for the components of the curvature tensor scalar curvature τ . Are then given by setting $\mathrm{P}_{ij} = R_{ijkk}$ *and* $\tau = \mathrm{P}_{ij} = R_{ikki}$. We shall often have an auxiliary vector bundle set V and an auxiliary given on V , we use this connection and the " Levi-Civita" connection to covariant differentiation , let dy be the measure of the induced metric on boundary ∂M , we choose a local orthonormal from near the boundary M , so that $\{e_m\}$ is the inward unit normal . We let indices a,b range from 1 to m-1 and index the induced local frame $\{e_1, \ldots\ldots, e_{m-1}\}$ for the tangent bundle f the boundary , let $L_{a,b} =_g \left(\nabla_{e_a} eb, e_m\right)$ denote the second fundamental form . we some over indices with the implicit range indicated . Thus the geodesic curvature K_g is given by $K_g = L_{aa}$. We shall let denote multiple tangential covariant differentiation with respect to the "Levi-Civita" connection the boundary the difference between and being of course measured by the fundamental form.

Proposition 2.3.1[Manifold admits a Riemannian Metric]

Any manifold a demits a Riemannian metric

Proof :

Take a converging by coordinate neighborhoods and a partition of unit subordinate to covering on each open set U_α we have a metric $g_\alpha = \sum_i dx_i^2$. In the local coordinates, define $g = \sum_i \varphi_i g_{\alpha(i)}$ this sum is well-defined because the support of φ_i . Are locally finite. Since $\varphi_i \geq 0$ at each point every term in the sum is positive definite or zero, but at least one is positive definite so that sum is positive definite.

Proposition 2.3.2[The Geodesic Flow]

Consider any manifold M and its cotangent bundle $T^*(M)$, with projection to the base $p : T^*(M) \to M$, let X be tangent vector to $T^*(M)$ at the point $\zeta \in T_a^* M$ then $D_p(X) \in T^*(M)$ so that $\varphi(X) = \zeta_a(D_p(x))$ defines a conicala conical 1-form φ on $T^*(M)$ in coordinates $(x, y) \to \sum_i y_i dy$ the projection p is $p(x, y) = x$ so if $X = \sum a_i \frac{\partial}{\partial x_i} + \sum b_i \frac{\partial}{\partial y_i}$ so if given take the exterior derivative $w = -d\varphi = \sum dx_i \wedge dy_i$ which is the canonical 2-from on the cotangent bundle it is non-degenerate, so that the map $X \to (i \times w)$ from the tangent bundle of $T^*(M)$ to its contingent bundle is isomorphism. Now suppose f is smooth function an $T^*(M)$ its derivative is a 1-form df .Because of the isomorphism a above there is a unique vector field X on $T^*(M)$ such that $df = (i \times w)$ from the g another function with vector field Y , then .

(12) $\quad Y(t) = df\ (Y) = i_Y \cdot i^i X^w = -i\ X^i Y^w = -(X)_g$

On a Riemannian manifold we shall see next there is natural function on $T^*(M)$. In fact a metric defines an inner on T^* as well as on T for the map $X \to g(X, -)$ defines an isomorphism form T to T^* then $g\left(\sum_j g_{ij} dx_j \times \sum_k g_{kl}\, dx_l\right) = g_{ik}$ which means that $g^*(dx_j, dx_k) = g^{ik}$ where g^{ik} denotes the matrix to g_{ik} we consider the function $T^*(M)$ defined by $H(\zeta_a) = g^*(\zeta_a, \zeta_a)$.

2.4 Maxima and Minima Lecture

Example 2.4.1

A Texas based company called (Hamilton'swares) sells baseball bats at a fixed price c . A field researcher has calculated that the profit the company makes selling the bats at the price c is $p(c) = \left(\frac{-1}{2000}c^4 + \frac{1}{5}c^3 - \frac{51}{2}c^2 + 1150\ c\right)$ at what price should the company sell their bats to make the most money. Intuitively what would we have to do solve this problem. We wish to know at what point c is this function $P(c)$ is maximized. we do not have many tools as moment to solve this problem so let's try to graph the function and guess at where the value should be.

Definition 2.4.2

Let f be function defined on an interval I containing c we say that f has an absolute maximum (or a global maximum) value on I at c $f(x) \leq f(c)$ for all x contained in I . Similarly, we say that f has an absolute minimum (or a global minimum) value on I at c if $f(x) \geq f(c)$ for all x contained in I . Those points together are known as absolute global extreme.

Example 2.4.3

$f(x) = x^2 + 1$ for $x \in (-\infty, \infty)$ remember this notation means for x living in the interval from negative infinity to infinity . This can also be written as $x \in R$ or in words as for all real x ,this function has an absolute minimum of at the point $x = 0$ but no absolute maximum on the interval .

Example 2.4.4

$f(x) = x^2 + 1$ for $x \in (-2,2)$ remember closed brackets means we include the endpoints in our interval this function has an absolute minimum of I at the point $x = 0$ and a absolute maximum of $f(\pm 2) = (\pm 2)^2 + 1 = 5$ at the points $x = 2$ and $x = -2$.

Example 2.4.5

$f(x) = x^2 + 1$ for $x \in (0,2)$ remember open brackets means we omit the endpoint in our interval .

Example 2.4.6

$f(x) = x^3$ for $x \in (-\infty, \infty)$, this function has no absolute minimum and no absolute maximum .

Definition 2.4.7[Extreme value theorem]

A function have an a absolute maximum and minimum these examples seen to suggest that if we have a closed interval then we're in business.

Example 2.4.8

Consider the function .from the graph, it's clear that this function has no absolute minimum or absolute maximum but $f(x)$ is defined on all of $[0,2]$ the problem with this example is that the function is not continuous.

(13) $$f(x)=\begin{cases} x & if \quad 0 \le x \le 1 \\ 1.5 & if \quad x = 0,1,2 \\ -x+4 & if \quad 1 \le x \le 2 \end{cases}$$

Theorem 2.4.9[Extreme value]

Let $f(x)$ be a continuous function defined on a close interval, then $f(x)$ has an absolute maximum and an absolute minimum on that interval .

[Notice]: that this says nothing about uniqueness. Remember the example $f(x)=x^2+1$ for $x\in[-2,2]$ has two points where the absolute maximum was obtained. Also note that functions that are not continuous and defined on a closed interval can still have extreme.

Example 2.4.10

Consider the following function on $[-1,1]$ as function $f(x)$, this function is not continuous at 0 however it has a global minimum of 0 of -3 because at all non-zero points this
function is sturdily positive.

(14) $$f(x)=\begin{cases} x & if \quad x \ne 0 \\ -3 & if \quad x = 0 \end{cases}$$

Definition 2.4.11

Let I be an open interval on which a function f is defined and suppose that $c\in I$ .We say that c is a local maximum value of f if $f(x)\le f(c)$ for all x contained in some open interval of I .Similarly we say that c is a local minimum value of f if $f(x)\ge f(c)$ for all x contained in some open interval I . These points together are known as local extreme.

[Note] : Your textbook uses any arbitrary interval, but requires c to be an interior point.

[Note] : Global extreme of a function that occur on an open interval contained in our domain are also local extreme.

Theorem 2.4.12 [Fermat's or local extreme]

If a function $f(x)$ has a local minimum or maximum at the point c and $f'(c)$ exists, then $f'(c)=0$

Example 2.4.13

We look at $f(x)=|x|$. Notice that this function is not differentiable $x=0$ but since $f(x)=|x|\succ 0 = f(0)$ we see that it has a local minimum at 0 (and in fact this is a global minimum).

Definition 2.4.14

A critical point is a point c in the domain of f where $f'(c)=0$ or $f'(c)$ fails to exist . In fact all critical points are candidates for extreme but it is not true that all critical points are extreme.

Example 2.4.15

Consider the function $f(x)=x^3$ .We saw before that this function has no maximum or minimum. However $f'(x)=3x^2$ and $f(0)=3(0)^2=0$ so the point $x=0$ is a critical point of f that is not an extreme.

2.5 : [Algorithm for finding global minima and maxima]

Let f be a continuous function on a closed interval $[a,b]$ so that our algorithm satisfies the conditions the conditions of the extreme value theorem:(i) Find all the critical points of (a,b), that is the points $x\in(a,b)$ where $f'(x)$ is not defined or where $f'(x)=0$ (usually done by setting the numerator and denominator to zero) call these points $x_1,x_2,....,x_n$.(ii) Evaluate $f(x_1),....., f(x_n), f(a), f(b)$ that is evaluate the function at all the critical points found from the previous step and the two end point values . (iii) The largest and the smallest values found in the previous step are the global minimum and global maximum values.

Example 2.5.1

Compute the absolute maximum and minimum of $3x-4x+2 \ on \ [-1,2]$.

Solution

Our function is continuous (and in fact differentiable) everywhere . Hence we $f'(x)=6x-4$ setting $f'(x)=0$ and solving yields $0=f'(x)=6x-4 \Rightarrow 4=6x \Rightarrow 2/3=x$. Now we evaluate f at $x=2/3,-1$ and 2 (that is the critical points and the end points) we get that .

(15) $$\left[f\left(\frac{2}{3}\right)=3\left(\frac{2}{3}\right)^2-4\left(\frac{2}{3}\right)+4=\frac{2}{3} \ , f(-1)=3(-1)^2-4(-1)+2=9 \ , f(2)=3(2)^2-4(2)+2=0 \right]$$

From this , we see that the absolute maximum is 9 obtained at $x=-1$ and the absolute minimum is $(2/3)$ obtained at $x=(2/3)$.

Example 2.5.2
Compute the critical points of $f(x) = 5x^{2/3}$.
Solution
We compute the derivative $f'(x) = \frac{10}{3} x^{\frac{-1}{3}}$,Now we check when the derivative is 0 and when it is undefined This function is never 0 but happens to be undefined at 0 which is a point in our in domain . Hence the critical points are just $x = 0$.

2.5: The Geometric of Operators of Laplace and Dirac Type
In this subsection we shall establish basic definitions discuss operator of Laplace and of Dirac type introduce the De-Ream complex and discuss the Boehner Laplacian and the weitzenbochformula. Let D be a second of smooth sections $C^{\infty}(v)$ of a vector bundle V over space M , expand $D = -\{ a^{\mu\nu}\partial x_{\mu}\partial x_{\nu} + a^{\sigma}\partial x_{\sigma} + b \}$ where coefficient $\{ a^{\mu\nu}, a^{\mu}, b \}$ are smooth endomorphism's of v, we suppress the fiber indices . We say that D is an operator of Laplace type if A^2 ,on $C^{\infty}(v)$ is said to be an operator of Dirac type if A^2 is an operator of Laplace operator of Dirac type if and only if the endomorphism's γ^{ν} satisfy the Clifford commutation relations $\gamma^{\nu}\gamma^{\mu} + \gamma^{\mu}\gamma^{\nu} = -2g^{\nu\mu}(id)$. Let A be an operator of Dirac type and let $\zeta = \zeta_{\nu}dx^{\nu}$ be a smooth 1-form on M we let $\gamma(\zeta) = \zeta_{\nu}v^{\nu}$ define a Clifford module structure on V . This is independent of the particular coordinate system chosen. We can always choose a fiber metric on so that γ is skew adjoin. We can then construct a unitary connection ∇ on V so that $\nabla\gamma = 0$ such that a connection is called compatible the endomorphism if ∇ is compatible we expand $A = \gamma\nabla^{\nu}_{\partial x_{\nu}} + \psi_A$, ψ_A is tonsorial and does not depend on the particular coordinate system chosen it does of course depend on the particular compatible connection chosen.
Definition 2.5.1 [The De-Rahm Complex]
The prototypical example is given by the exterior algebra, let $C^{\infty}(\Lambda^{p}M)$ be the space of smooth p forms. Let $d : C^{\infty}(\Lambda^{p}M) \to C^{\infty}(\Lambda^{p-1}M)$ be exterior differentiation if ζ is cotangent vector, Let $ext(\zeta) : w \to \zeta \wedge w$ denote exterior multiplication and let $\mathrm{int}(\zeta)$ be the Dual , Interior multiplication, $v(\zeta) = ext(\zeta) - \mathrm{int}(\zeta)$ define module on exterior algebra $\Lambda(M)$. Since $d + \delta = v(dx^{\nu})\nabla_{\partial x_{\nu}}$. $d + \delta$ is an operator of " Direct type" the a associated Laplacian $\Delta_m = (d+\delta)^2 = \Delta^0_m \oplus ...... \oplus \Delta^p_m \oplus ..... \oplus \Delta^m_M$ decomposes as the "Direct sum" of operators of Laplace type Δ^p_m on the space of smooth p forms $C^{\infty}(\Lambda^{p}M)$ on has $\Delta^0_M = -g^{-1}\partial x_{\mu} g\, g^{\mu\nu}(\partial x_{\nu})$ it is possible to write the p-form valued Laplacian in an invariant form . Extend the " Levi-Civita" conduction to act on tensors of all types .Let $\tilde{\Delta}_{M^{\nu}} = -g^{\mu\nu}w$, $\mu\nu$ define Buchner or reduced Laplacian , let R given the associated action of curvature tensor . The "Weitzenbock" formula terms of the"Buchner Laplacian" in the form $\Delta_M = \tilde{\Delta}_M + \frac{1}{2}\gamma(dx^{\mu})\gamma(dx^{\gamma})R_{\mu\gamma}$
This formalism can be applied more generally.
Lemma 2.5.2[Spinner Bundle]
Let D be an operator of Laplace type on a Riemannian manifold, there exists a unique connection ∇ on V and there exists a unique endomorphism E of V , so that $D\phi = -\phi_{ii} - E\phi$ if we express D locally in the form $D = \{ g^{\mu\gamma}\partial x_{\gamma}\partial x_{\mu} + a^{\mu}\partial x_{\mu} + b \}$ then the connection 1-form w of ∇ and the endomorphism E are given by .

$$(16) \qquad \left[w_{\gamma} = \frac{1}{2}\left(g_{\gamma\mu}a^{\mu} + g^{\sigma E}\Gamma_{\sigma E_{\gamma}} id \right) \text{ and } E = b - g^{\gamma\mu}\left(\partial x_{\gamma} w_{\mu} + w_{\gamma}w_{\mu} - w_{\sigma}\Gamma^{\sigma}{}_{\gamma\mu} \right) \right]$$

Let V be equipped with an auxiliary fiber metric, then D is self-adjoin if and only if ∇ is unitary and E is self-adjoin we note if D is the Spinner bundle and the "Lichnerowicz formula" with our sign convention that $E = -\frac{1}{4}J(id)$ where J is the scalar curvature.
Definition 3.4.3 Heat Trace Asymptotic for closed manifold
Throughout this section we shall assume that D is an operator of Laplace type on a closed Riemannian manifold (M, g) . We shall discuss the L^2 - spectral resolution if D is self-adjoin , define the heat equation introduce the heat trace and the heat trace asymptotic present the leading terms in the heat trace . Asymptotic references for the material of this section and other references will be cited as needed , we suppose that D is self-adjoin there is then a complete spectral resolution of D on $L^2(v)$. This means that we can find a complete orthonormal basis $\{\phi_n\}$ for $L^2(v)$ where they ϕ_n are a smooth sections to V which satisfy the equation $D\phi_n = \lambda_n\phi_n$.

2.6 : Inverse Spectral Problems in Riemannian geometry

In al-arguably one the simplest inverse problem in pure mathematics " can on hear the shape of drum " mathematically the question is formulated as follows , let Ω be a simply connected plane domain (The drumhead bounded by a smooth curve γ) , and consider the wave equation on Ω with . Dirichlet boundary condition on γ - (the drumhead is clamped at boundary)

$$\Delta u(x,t) = \frac{1}{C^2} U_{tt}(x,t) \quad in \ \Omega \qquad , \quad U(x,t) = 0 \quad in \ \gamma$$

The function $U(x,t)$ is the displacement of drumhead as vibrates at position x at time t , looking for solutions of the form $U(x,t) = \operatorname{Re} \ e^{iwt} v(x)$ (normal modes) leads to an eigenvalue problem for the Dirichlet Laplacian on Ω Where $\lambda = w^2 / c^2$, we write the infinite sequins of Dirichlet eigenvalues for this problem as $\{\lambda_n(\Omega)_{n=1}^{\infty}\}$ or simply $\{\lambda_n\}_{n=1}^{\infty}$, if the choice of domain Ω is clear in context , Kans question means the following is it possible to distinguish " drums " Ω_1 and Ω_2 with distinct (modulo isometrics) bounding curves γ_1 and γ_2 simply by (hearing) all of the eigenvalues of Dirichlet Laplacian some surprising and interesting results are obtained by considering the heat equation on Ω with
Dirichlet boundary conditions, which given rise to the same boundary value
problem as before the heat equation is :

(17)
$$\begin{cases} \Delta U(x,t) = U_t(x,t) & in \ \Omega \\ U(x,t) = 0 & on \ \gamma \\ U(x,0) = f(x) & \end{cases}$$

Where $U(x,t)$ is the temperature at point x and time t , and f(x) is the initial temperature distribution. This evolution equation is formal solution. $U(x,t) = \left(e^{t\Delta} f\right)(x)$. Where the operator $e^{t\Delta}$ can be calculated using the spectral resolution of Δ . Indeed if $\phi_j(x)$ is the normalized Eigen function of the boundary value problem with eigenvalue λ_j the operator $e^{t\Delta}$ has integral Kernel $k(t,x,y)$ the heat Kernelgiven by .

(18)
$$k(t,x,y) = \sum_{j=1}^{\infty} e^{-t\lambda} \phi(x) \ \phi(y)$$

The trace of $k(t,x,y)$ is actually a spectral in variant by (we can compute).

$$k(t,x,y) = \sum_{j=1}^{\infty} e^{-t\lambda_j}$$

[Not] that the function determines the spectrum $\{\lambda_n\}_{n=1}^{\infty}$, to analyze the geometric content of spectrum, one calculates the by completely different method one constructs the heat kernel by perturbation from the explicit heat kernel for the plane, and then on computes the trace explicitly. It turns out that the trace has a small-t asymptotic expansion.

(19)
$$\int_{\Omega} k(x,x,t) \, dx \approx \frac{1}{4\pi t}\left(a_0 + a_{1/2} + a_1 t + ... \right)$$

Where $a_0 = area \ (\Omega)$, $a_1 = length \ (\gamma)$, Al though a strict derivation is a bit involved which shows why a_0 and a_1 should given the area of Ω and length of γ the heat kernel in the plan is . $k_0(x,y,t) = \frac{1}{4\pi t} \exp\left(\left|x-y\right|^2 / 4\pi t \right)$, we expect particle that for small times $K(x,x,t) \approx k_0(x,x,t)$ (a Brownian particle starting out the interior doesn't the boundary for a time of order $\sqrt{t}$).

(20)
$$\int_{\Omega} K(x,x,t) \ dx \approx \int_{\Omega} k_0(x,x,t) dx = \frac{1}{4\pi t} \ are \ (\Omega)$$

For times of order $\sqrt{t}$, boundary effects become important we can approximate the heat kennel near the boundary locally by (method images) locally the boundary looks the line $x_1 = 0$ *in the* $x_1 - x_2$ plane , letting $x \to x^*$ be , $K_2(x,x,t) = k_0(x,x,t) - k(x,y,t)^*$ vanishes $x_1 = 0$ hence $K_{\Omega}(x,x,t) \approx \frac{1 - e^{-2\delta^2/t}}{4\pi t}$ where δ . Is the distance from x to the boundary , writing the volume integral for the additional term as an integral over the boundary curve and distance from the boundary $\int_{\gamma} \int_0^{\infty} \frac{1}{4\pi t} e^{-2\delta/t} d\delta \ ds$ we have.

(21)
$$\int_{\Omega} K(x,x,t) \, dx \approx \frac{area \ (\Omega)}{4\pi t} - \frac{length \ (\gamma)}{4} \frac{1}{\sqrt{2\pi t}} + \left(\frac{1}{\sqrt{t}}\right)$$

It follows that the is spectral set of a given (drum) Ω contains only drums with the same area and perimeter here we will briefly discuss the generalization of kais problem and some of the known results. A Riemannian manifold of dimension n is a smooth n-dimensional manifold M . Equipped with a Riemannian metric g which defines the length of tangent vectors and determines distances and angles on the manifold . The metric also determines the Riemann curvature tensor of M . In two dimensions , the Riemannian curvature tensor is in turn determined by the scalar curvature, and in three dimensions it is completely determined by the Ricci curvature tensor. If M is compact the associated Laplacian has infinite set of discrete eigenvalues $\{\lambda_n\}_n^{\infty} = 1$ what is the geometric content of the spectrum for a compact Riemannian manifold. Constructs a pair 16-dimensional tori with with the same spectrum. The torii T_1^n *and* T_2^n are quotients of R^n by lattices Γ_1 *and* Γ_2 of translations of R^n. Since the tow tori are isometric of and only if their lattices are congruent, it suffices to construct a pair of non-congruent 16-dimensional lattices whose a associated tori have the same spectrum .To understand the analysis involved in Milnor's construction consider the following simple " trace formula" for a torus $T^n = R^n / \Gamma$ which computes the trace of the heat kernel on a torus in terms of lengths of the lattice vectors to the heat kernel on the torus is given by the formula. $K_{\Gamma}(x,x,t) = \sum_{w\in\Gamma} k_0(x+w,y,t)\ K_0(x,x,t)dx = \frac{vol\,(T^n)}{4\pi t}\sum_{w\in\Gamma} e^{-|w|^2/4t}$

Milnor noted that there exist non-congruent lattices in 1-dim. With the same set of " length "$\{|w| : w\in\Gamma\}$ first discovered by the trace of the heat kernel determines the spectrum and the heat trace is in turn determined by the lengths, it follows that the corresponding non-isometric tori have the same spectrum.

Example 2.6.1 [Riemannian Manifold with Same Spectrum]

Riemannian manifold with the same spectrum letter constructed continuous families of is spectral manifold in sufficiently high dimension $n \geq 5$ two major questions remained:

(i) can one show that the is spectral set of given manifold at finite in low dimension .

(ii) can on find counterexamples for Kicks original problem , can one construct is spectral , non-congruent planar

Definition 2.6.2 [Some Positive Results]

In proved one of the first major positive results on is spectral sets of surfaces and planar domains informally. A sequence of planar domains Ω_j converges in C^{∞} since to a limiting non-degenerate set compact surfaces S_j converges in C^{∞} sense to limiting non-degenerate surface S , converge in C^{∞} sense to a positive definite metric on S

Theorem 2.6.3

(i) Let Ω_j be a sequence of is spectral planar domains there is a subsequence which converges in C^{∞} sense to no degenerate limiting surface.

(ii) Let S_j be a sequence of is spectral compact surfaces there is a subsequence of the S_j converging in C^{∞} sense to a non-degenerate surface S .

Theorem 2.6.4

Suppose $\{M_j\}$is a sequence if is spectral manifold such that either : (i) All of the M_j- have negative sectional curvatures .(ii) All of the M_j -have Ricci curvatures bounded below .Then there are finitely many diffoemorphism types and there is a subsequence which convergent in C^{∞} to a nodegenerate limiting manifold .

(22) $$\frac{d\gamma}{dt}(t) = \left(\frac{d\gamma^1}{dt}t_0,\ldots,\frac{d\gamma^n}{dt}t_0\right)$$

we many k bout smooth curves that is curves with all continuous higher derivatives cons the level surface $f(x^1,x^2,\ldots,x^n) = c$ of a differentiable function f where x^i to $(i-th)$ coordinate the gradient vector of f at point $P = x^1(P), x^2(P),\ldots, x^n(P)$ is $\nabla f = \left(\frac{\partial f}{\partial x^1},\ldots,\frac{\partial f}{\partial x^n}\right)$ is given a vector $u = (u^1,\ldots,u^n)$ the direction derivative $D_u f = \nabla f\cdot\overline{u} = \frac{\partial f}{\partial x^1}u^1 + \ldots + \frac{\partial f}{\partial x^n}u^n$, the point P on level surface $f(x^1,x^2,\ldots,x^n)$ the tangent is given by equation. $\frac{\partial f}{\partial x^1}(P)(x^1 - x^1)(P) + \ldots + \frac{\partial f}{\partial x^n}(P)(x^n - x^n)(P) = 0$

For the geometric views the tangent space shout consist of all tangent to smooth curves the point P , assume that is curve through $t = t_0$ is the level surface. $f(x^1,x^2,\ldots,x^n) = c$, $f(\gamma^1(t),\gamma^2(t),\ldots,\gamma^n(t)) = c$ by taking derivatives on both $\frac{\partial f}{\partial x^1}(P)(\gamma'(t_0) + \ldots + \frac{\partial f}{\partial x^n}(P)\gamma^n(t)) = 0$ and so the tangent line of γ is really normal orthogonal to ∇f , where γ runs over all possible curves on the level surface through the point P . The surface

M be a C^{∞} manifold of dimension n with $k \geq 1$ the most intuitive to define tangent vectors is to use curves , $p \in M$ be any point on M and let $\gamma :]-\varepsilon, \varepsilon[\to M$ be a C^1 curve passing through p that is with $\gamma(M) = p$ unfortunately if M is not embedded in any R^N the derivative $\gamma'(M)$ does not make sense ,however for any chart (U, φ) at p the map $(\varphi \circ \gamma)$ at a C^1 curve in R^n and tangent vector $v = (\varphi \circ v)\ (M)$ is will defined the trouble is that different curves the same v given a smooth mapping $f : N \to M$ we can define how tangent vectors in $T_p N$ are mapped to tangent vectors in $T_q M$ with (U, φ) choose charts $q = f(p)$ for $p \in N$ and (V, ψ) for $q \in M$ we define the tangent map or flash-forward of f as a given tangent vector.

(23) $X_p = [\ \gamma\ ] \in T_p N$ and $d\ f_* : T_p M \quad , \quad f_*([\ \gamma\ ]) = [\ f \circ \gamma\ ])$

A tangent vector at a point p in a manifold M is a derivation at p , just as for R^n the tangent at point p form a vector space $T_p(M)$ called the tangent space of M at p , we also write $T_p(M)$ a differential of map $f : N \to M$ be a C^{∞} map between two manifolds at each point $p \in N$ the map F induce a linear map of tangent space called its differential p , $F_* : T_p N \to T_{F(p)} N$ as follows it $X_p \in T_p N$ then $F_*(X_p)$ is the tangent vector in $T_{F(p)} M$ defined .

(24) $\left(F_*(X_p)\right) f = X_p(\ f \circ F\ ) \in R, \quad f \in C^{\infty}(M)$

The tangent vectors given any C^{∞} - manifold M of dimension n with $k \geq 1$ for any $p \in M$,tangent vector to M at p is any equivalence class of C^1 - curves through p on M modulo the equivalence relation defined in the set of all tangent vectors at p is denoted by $T_p M$ we will show that $T_p M$ is a vector space of dimension n of M . The tangent space $T_p M$ is defined as the vector space spanned by the tangents at p to all curves passing through point p in the manifold M , and the cotangent $T_p^* M$ of a manifold at $p \in M$ is defined as the dual vector space to the tangent space $T_p M$, we take the basis vectors $E_i = \left(\frac{\partial}{\partial x^i}\right)$ for $T_p M$ and we write the basis vectors $T_p^* M$ as the differential line elements $e^i = dx^i$ thus the inner product is given by.

(25) $\left\langle \partial / \partial x, dx^i \right\rangle = \delta_i^j$

Theorem 2.6.5[Bounded Harmonic Function]

Suppose that Ω is a bounded, connected open set in R^n and $U \in C^2(\Omega) \cap C(\overline{\Omega})$ is harmonic in Ω then.

$\max_{\Omega} U = \max_{\partial\Omega} U$ and $\min_{\Omega} U = \min_{\partial\Omega} U$

Proof :

Since U is continuous and $\overline{\Omega}$ is compact , U attain its global maximum and minimum on $\overline{\Omega}$, if U attains a maximum or minimum value at interior point then U is constant by otherwise both extreme values are attained in the boundary .In either cases the result follows let given a second of this theorem that does not depend on the mean value property .Instated we us argument based on the non-positivity of the second derivative at an interior maximum . In the proof we need to account for the possibility of degenerate maxima where the second derivative in zero . For $\varepsilon \geq 0,\ let\ U^{\varepsilon}(x) = U(x) + \varepsilon\left|x\right|^2$. Then $\Delta U^{\varepsilon} = 2n \quad \varepsilon \geq 0$,since U is harmonic .if U^{ε} attained a local maximum at an interior point then $\Delta U^{\varepsilon} \leq 0$ by the second derivative test thus U^{ε} no interior maximum, and it attains its maximum on the boundary .If, $\left|x\right| \leq R,\ for\ all\ x \in \Omega$, if follows that.

(26) $Sup_{\Omega} U \leq Sup_{\Omega} U^{\varepsilon} \leq Sup_{\partial\Omega} U^{\varepsilon} \leq Sup_{\partial\Omega} U + \varepsilon R^2$

letting $\varepsilon \to 0^+$,we get that $Sup_{\Omega} U \leq Sup_{\partial\Omega} U$ .An application for the same a grummet to u given in, $\inf_{\Omega} U \leq \inf_{\partial\Omega} U$ .and the result follows . Sub harmonicfunction satisfy a maximum principle $\min_{\overline{\Omega}} U = \min_{\partial\Omega} U$,while sub harmonic function satisfy a minimum principle $\min_{\overline{\Omega}} U \leq U \leq \min_{\partial\Omega} U$ for all $x \in \Omega$. Physical terms, this means for example that the interior ofabounded region which contains no heat sources on heat sources or sinks cannot be hotter that the maximum temperature on the boundary or colder than the minimum temperature on the boundary .The maximum principle given a uniqueness result for (Dirichlet problem) for the poison equation .

Definition 2.6.6

Let M_1 and M_2 be differentiable manifolds a mapping $\varphi : M_1 \to M_2$ is a differentiable if it is differentiable objective and its inverse φ^{-1} is diffoemorphism if it is differentiable φ is said to be a local diffoemorphism at $p \in M$ if there exist neighborhoods U of p and V of $\varphi(p)$ such that $\varphi : U \to V$ is a diffoemorphism , the notion of diffoemorphism is the natural idea of equivalence between differentiable manifolds , its an immediate consequence of the chain rule that if $\varphi : M_1 \to M_2$ is a diffoemorphism then.

(27) $$d\varphi : T_p M_1 \to T_{\varphi(p)} M_2$$

Is an isomorphism for all $\varphi : M_1 \to M_2$ in particular , the dimensions of M_1 and M_2 are equal a local converse to this fact is the following $d\varphi : T_p M_1 \to T_{\varphi(p)} M_2$ is an isomorphism then φ is a local diffoemorphism at p from an immediate application of inverse function in R^n , for example be given a manifold structure again A mapping $f^{-1} : M \to N$ in this case the manifolds N and M are said to be homeomorphism , using charts (U, φ) and (V, ψ) for N and M respectively we can give a coordinate expression $\tilde{f} : M \to N$

Theorem 2.6.7 [Compact Riemannian Manifolds]

Let M be compact Riemannian Manifold with or without boundary ∂M , $f(M) \in C(F, R)$. proper function satisfying. $\beta(r - s) \leq F(x, r, p, X) - F(x, r, p, X) \quad for \quad r \geq s$ There exist a function $W : [0, \infty] \to [0, \infty]$. satisfying $W(t) \geq 0 \;\; when \;\; t \geq 0$ and $W(0+) = 0$.

Definition 2.6.8

Let M_1^{-1} and M_2^{-1} be differentiable manifolds and let $\varphi : M_1 \to M_2$ be differentiable mapping for every $p \in M_1$ and for each $v \in T_p M_1$ choose a differentiable curve $\alpha : (-\varepsilon, \varepsilon) \to M_1$ with $\alpha(M) = p$ and $\alpha'(0) = v$ take $\alpha \circ \beta = \beta$ the mapping $d\varphi_p : T_\varphi(p) M_2$ by given by $d\varphi(v) = \beta'(M)$ is line of α and $\varphi : M_1^{-1} \to M_2^{-1}$ be a differentiable mapping and at $p \in M_1$ be such $d\varphi : T_p M_1 \to T_\varphi M_2$ is an isomorphism then φ is a local homeomorphism .

Proposition2.6.9

Let M_1^n and M_1^m be differentiable manifolds and let $\varphi : M_1 \to M_2$ be a differentiable mapping , for every $p \in M_1$ and for each $v \in T_p M_1$ choose a differentiable curve $\alpha : (-\varepsilon, \varepsilon) \to M_1$ with $\alpha(o) = p$, $\alpha'(o) = v$ take $\beta = \varphi \circ \alpha$ the mapping $d\varphi : T_p M_1 \to T_{\varphi(p)} M_2$ given by $d\varphi_p(v) = \beta'(o)$ is a linear mapping that dose not depend on the choice of α .

Theorem 2.6.10

The tangent bundle TM has a canonical differentiable structure making it into a smooth 2n-dimensional manifold, where N=dim. The charts identify any $U_p \in U(T_p M) \subseteq (TM)$ for an coordinate neighborhood $U \subseteq M$, with $U \times R^n$ that is Hausdorff and second countable is called (The manifold of tangent vectors)

Definition 2.6.11

A smooth vectors fields on manifolds M is map $X : M \to TM$ such that:(i) $X(P) \in T_p M$ for every G .(ii) in every chart X is expressed as $a_i(\partial / \partial x_i)$ with coefficients $a_i(x)$ smooth functions of the local coordinates x_i .

Theorem2.6.12

Suppose that on a smooth manifold M of dimension n there exist n vector fields $\left(x^{(1)}, x^{(2)}, ..., x^{(n)}\right)$ for a basis of $T_p M$ at every point p of M , then $T_p M$ is isomorphic to $M \times R^n$ m here isomorphic means that TM and $M \times R^n$ are homeomorphism as smooth manifolds and for every $p \in M$, the homeomorphism restricts to between the tangent space $T_p M$ and vector space $\{P_i\} \times R^n$.

Proof:

define $\pi : \vec{a} \in T_p M \subset TM$ on other hand , for any $M \times R^n$ for some $a_i \in R$ now define $\Phi : \vec{a} \in TM \to \left(\pi(s) : a_1, ..., a_n \in M \times R^n\right)$ is it clear from the construction and the hypotheses of theorem that Φ and Φ^{-1} are smooth using an arbitrary chart $\varphi : U \subseteq M \to R^n$ and corresponding chart.

(28) $$\varphi T : \pi^{-1}(U) \subseteq TM \to R^n \times R^m$$

Definition 2.6.13 [Direct Computationof The Spectrum]

The first of those is straightforward: direct computation it rarely possible to explicitly compute the spectrum of a manifold were actually discovered via this method . Milnor's example mentioned above consists of two is spectral factory-quotients of Euclidean space by lattices of full rank being one of full rank being one of the few examples of Riemannian manifolds whose spectra can be computed explicitly spherical space forms – quotients of spheres by finite groups of orthogonal transformations acting without fixed points form another class of examples of manifolds is spectral for the Laplacian acting on p-forms for $p \leq k$ but not for the

Laplacian acting on p-forms for $p \le k+1$ (recall that a lens space is spherical space form where the group is cyclic .Definition

3.5.14 Tensors on A vector Space

A tensor ϕ on V is by definition a multiline map .

(29) $\phi : \left\{ \underbrace{V \times ..... \times V}_{r} \times \underbrace{V^{*} \times ..... \times V}_{s}{}^{*} \to R \right\}$

V^* denoting the dual space to V , r its covariant order, and s its contra variant order.(Assume $r > 0$ or $s > 0$). Thus ϕ assigns to each r-tuple of elements of V and s-tupelo of elements of V^* a real number and if for each k, $1 \le k \le r+s$, we hold every variable except the k-th fixed, then ϕ Satisfies the linearity condition .

(30) $\phi(v_1,..., \alpha\ v_k + \alpha' v'_k,...) = \alpha\phi\ (v_1,..., v_k,...) + \alpha'\phi(v'_1,..., v'_k,..._1)$.

For all $\alpha, \alpha' \in R$ and $v_k, v'_k \in V$ or (V^*) .respectively For a fixed (r,s) we let $^r_s(V)$ be the collection of all tensors on V of covariant order r and contra variant order s. We know that as a function from $V \times .... \times V \times V^* \times ... \times V^*$ to R they may be added and multiplied by scalars elements of R. With this addition and scalar Multiplication $(V)^r_s$ is a vector space, so that if $\phi_1, \phi_2 \in {}^r_s(V)$ and $\alpha_1, \alpha_2 \in R,$ then $\alpha_1\phi_1 + \alpha_2\phi_2$, defined in the way alluded to above, that is, by.

(31) $\begin{cases} (\alpha_1\phi_1 + \alpha_2\phi_2) \qquad (v_1, v_2,....) \\ = \alpha_1\phi_1(v_1, v_2,....) + \alpha_2\phi_2(v_1, v_2,....) \end{cases}$

is multiline, and therefore is in $(V)^r_s$. Thus $(V)^r_s$ has a natural vector space structure.

Theorem 2.6.15

With the natural definitions of addition and multiplication by elements of R the set $(V)^r_s$ of all tensors of order (r,s) on V forms a vector space of dimension n^{r+s} .

Definition 2.6.16 [Tensor Fields]

A C^∞ covariant tensor field of order r on a C^∞ - manifold M is a function ϕ which assigns to each $P \in M$ an element ϕ_P of $(T_P(M))^r$ and which has the additional property that given any C^∞ – Vector fields on an open subset U of M , then $\phi(X_1,...., X_r)$ is a C^∞ function on U , defined by , $\phi(X_1,...., X_r)\ (P) = \phi_P(X_{1P},....., X_{rP})$. We denote by $(M)^r$ the set of all C^∞ – covariant tensor fields of order r on M .

Definition 2.6.17

We shall say that $\phi \in V^r$, $\phi \in V^r$ a vector space, is symmetric if for each $1 \le i, j \le r$, we have : $\phi(v_1,..., v_j,..., v_i,..., v_r) = \phi(v_r,..., v_j,..., v_i,..., v_1)$ .Similarly, if interchanging the (i-th) and (j-th) variables, $1 \le i, j \le r$ Changes the sign.

(32) $\phi(v_1,..., v_j,..., v_i,..., v_r) = -\phi(v_r,..., v_j,..., v_i,..., v_1)$

then we say ϕ is skew or anti symmetric or alternating; covariant tensors are often called exterior forms. A tensor field is symmetric (respectively, alternating) if it has this property at each point.

2.7: Geometrid Maximum and principle Riemannian manifolds

The version of the analytic principle given by:

(i) U_0 is lower semi – continuous and $M[U_0] \le H_0$ in the sense of support function.(ii) U_1 is upper – semi – continuous and $M[U_0] \ge H_0$ in the sense function with a one – sided Hessian bound .(iii) $U_1 \le U_0$ in Ω and $U_1 = U_0$ is locally a $C^{1,1}$ - function in Ω finally if a^{ij} and b are locally $C^{k+2,\alpha}$ function in Ω . In particular if a^{ij} and b are smooth is $U_1 = U_0$, $\Omega \subset R^n$ is specially natural in Lorentz Ian setting as C^0 space like hyper surfaces in definition $S_{\eta,r} = \{\ p : d\ (p, \exp\ \ (r,\eta)\ = r)\}$ them $S_{\eta,r}$ contains $\pi(\eta)$ and neighborhood of $\pi(\eta)$ is smooth , at $\pi(\eta)$ pointing unit normal $r \ge 0$ and $k \subset T(M)$ can a lows be locally represented as a graphs also applies to hyper surfaces in Riemannian manifolds that can be represented locally as graphs. We first state our conventions on the sign of the second fundamental form and the mean curvature to fix choice of signs a Lorentz Ian manifold $(M.g)$.

Definition 2.7.1[Space time and Space like]

A subset $N \subset M$ of that space-time $(M.g)$ is C^0 space likehyper surface , if for each $p \in N$, there is a neighborhood U of p in M so that $N \cap V$ is causal and edge less in U .

Remark 2.7.2
In This definition not that if $D(N \cap U, U)$ is the domain of dependence of in U , then $D(N \cap U, U)$ is open in M and $U \cap U$ is a Cauchy hyper surface is globally hyperbolic thus by replacing U by $D(N \cap U, U)$ we can assume the neighborhood U is the last definition is globally hyperbolic and that $U \cap U$ in a Cauchy surface in U . In particular a C^0 space like hyper surface is a topological.Let $(M.g)$ be a space-time and let N_0 and N_1 be two C^0 space like hyper surfaces in $(M.g)$ which meet at a point q . Say that no is locally to the future of N_1 near q if for some neighborhood U of P in which N_1 is a causal and edgeless $N_0 \cap U \subset J^+(N_1, \cup)$ where $J^+(N_1, \cup)$ is causal future of N_1 in U .

Definition 2.7.3 [Multiplication of Tensors on Vector Space]
Let V be a vector space and $\varphi \in V$ are tensors. The product of φ and ψ , denoted $\varphi \otimes \psi$ is a tensor of order $r+s$ defined by $\varphi \otimes \psi(v_1, ... v_r, ... v_{r+1}, ... v_{r+s}) = \varphi(v_1, ..., v_r)\psi(v_{r+1}, ..., v_{r+s})$.
The right hand side is the product of the values of φ and ψ .The product defines a mapping $(\varphi, \psi) \to \varphi \otimes \psi$ of x $^r(V) \to {}^{r+s}(V)$.

Theorem 2.7.4
The product $^r(V)$ o $^r(V) \to {}^{r+s}(V)$ just defined is bilinear and associative. If $\omega^1, ..., \omega^n$ is a basis of .

(33) $$V^* = \varphi_1 \wedge \varphi_2 \wedge .... \wedge \varphi_k = \frac{(r_1 + r_2 + ... + r_k)!}{r_1! \, r_2! .... r_k!}$$

then $\{ (\omega^{i_1} \otimes .... \otimes \omega^{i_r}) / (1 \le i_1, ..., i_r \le n) \}$ is a basisof $(V)^r$ .Finally $F_* : W \to V$ is linear, then $F^*(\varphi \otimes \psi) = (F^*\varphi) \otimes (F^*\psi)$.

Proof:
Each statement is proved by straightforward computation. To say that $\otimes$ is bilinear means that if α, β are numbers and $\psi \in {}^r(V)$, then $(\alpha\varphi_1 + \beta\varphi_2) \otimes \psi = \alpha(\varphi_1 \otimes \psi) + \beta(\varphi_2 \otimes \psi)$. Similarly for the second variable. This is checked by evaluating each Side on $r+s$ vectors of V ; in fact basis vectors suffice because of linearity Associatively, $(\varphi \otimes \psi) \otimes \theta = \varphi \otimes (\psi \otimes \theta)$,is similarly verified the products on both sides being defined in the natural way. This allows us to drop the parentheses. To see that $\omega^{i_1} \otimes ..... \otimes \omega^{i_r}$ from a basis it is sufficient to note that if $e_1, ..., e_n$ is the basis of V dual to $\omega^1, ..., \omega^n$, then the tensor $\Omega^{i_1, ..., i_r}$ previously defined is exactly $\omega^{i_1} \otimes ..... \otimes \omega^{i_r}$ .This follows from the two definitions:

(36) $$\Omega^{i_1 ... i_r}(e_{j_1}, ..., e_{j_r}) = \begin{Bmatrix} 0 & if\ (i_1, ..., i_r) \ne (j_1, ..., j_r) \\ 1 & if\ (i_1, ..., i_r) = (j_1, ..., j_r) \end{Bmatrix}$$

$(\omega^{i_1} \otimes .... \otimes \omega^{i_r})\ (e_{j_1}, ..., e_{j_r}) = \omega^{i_1}(e_{j_1})\omega^{i_2}(e_{j_2}), ..., \omega^{i_r}(e_{j_r}) = \delta^{i_1}_{j_1}\delta^{i_2}_{j_2}...\delta^{i_r}_{j_r}$,which show that both tensors have the same values on any (ordered) set of r basis vectors and are thus equal. Finally, given $F_* : W \to V$, if $w_1, ..., w_{r+s} \in W$, then

(35) $(F^*(\varphi \otimes \psi))(w_1, ..., w_{r+s}) = \varphi \otimes \psi\ (F_*(w_1), ..., F_*(w_{r+s})) = \varphi(F_*(w_1), ..., F_*(w_r))\ \psi(F_*(w_{r+1}), ..., F_*(w_{r+s})) = (F^*\varphi) \otimes (F^*\psi)(w_1, ..., w_{r+s})$.

Which proves $F^*(\varphi \otimes \psi) = (F^*\varphi) \otimes (F^*\psi)$ and completes the proof.

Theorem 2.7.8 [Multiplication of Tensor Field on Manifold]
Let the mapping $(M)^r \times (M)^s \to (M)^{r+s}$ just defined is bilinear and associative. If $(\omega^1, ..., \omega^n)$ is a basis of $(M)^1$, then every element $(M)^r$ is a linear combination with C^∞ coefficients of $\{(\omega^{i_1} \otimes .... \otimes \omega^{i_r}) / (1 \le i_1, ..., i_r \le n)\}$. If $F : N \to M$ is a C^∞ mapping, $\varphi \in M$ and $\psi \in {}^s(M)$, then $F^*(\varphi \otimes \psi) = (F^*\varphi)(F^*\psi)$, tensor field on N .

Proof:
Since two tensor fields are equal if and only if they are equal at each point, it is only necessary to see that these equations hold at each point, which follows at once from the definitions and the preceding theorem .

Corollary 2.7.9
Each $\varphi \in U^r$ including the restriction to U of any covariant tensor field on M , has a unique expression form $\varphi = \sum_{i_1} .... \sum_{i_r} a_{i_1 .... i_r}(\omega^{i_1} \otimes .... \otimes \omega^{i_r})$. Where at each point U , $a_{i_1 .... i_r} = \varphi(E_{i_1}, ..., E_{i_r})$ are the Components of φ in the basis $\{\omega^{i_1} \otimes ..... \otimes \omega^{i_r}\}$ and is C^∞ function on U .

2.8 : Tangent Space and Cotangent Space

The tangent space $T_p(M)$ is defined as the vector space spanned by the tangents at p to all curves passing through point p in the manifold M . And The cotangent space $T_p^*(M)$ of a manifold, at $p \in M$ is defined as the dual vector space to the tangent space $T_p(M)$. We take the basis vectors $E_i = \partial/\partial x^i$ for $T_p(M)$, and we write the basis vectors for $T_p^*(M)$ as the differential line elements $e^i = dx^i$ Thus the inner product is given by $\langle \partial/\partial x^i, dx^i \rangle = \delta_i^j$.

Definition 2.8.1 [Wedge Product]

Carton's wedge product, also known as the exterior Product, as the ant symmetric tensor product of cotangent space basis elements.

(36) $dx \wedge dy = \frac{1}{2}(dx \otimes dy - dy \otimes dx) = (-dy \wedge dx)$

Note that, by definition, $dx \wedge dx = 0$. The differential line elements dx and dy are called differential 1-forms or 1-form; thus the wedge product is a rule for construction g 2-forms out of pairs of 1-forms.

Definition 2.8.2

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Definition 2.8.3 [Vector Analysis one Method Lengths]

Classical vector analysis describes one method of measuring lengths of smooth paths in R^3 if $v : [0,1] \to R^3$ is such a paths, then its length is given by length $v = v(t)\, dt$. Where [v] is the Euclidean length of the tangent vector (t) , we want to do the same thing on an abstract manifold ,and we are clearly faced with one problem , how do we make sense of the length $|v(t)|$,obviously , this problem can be solved if we assume that there is a procedure of measuring lengths of tangent vectors at any point on our manifold The simplest way to do achieve this is to assume that each tangent space is endowed with an inner product (which can vary point in a smooth).

Definition 2.8.3

A Riemannian manifold is a pair $(M \,.g)$ consisting of a smooth manifold M and a metric g on the tangent bundle ,i.ea smooth symmetric positive definite tensor field on M . The tensor g is called a Riemannian metric on M . Two Riemannian manifold are said to be isometric if there exists a diffoemorphism $\phi : M_1 \to M_2$ such that $\phi^* : g_1 = g_2$ If $(M \,.g)$ is a Riemannian manifold then, for any $x \in M$ the restriction $g_x : T_x(M_1) \times T_x(M_2) \to R$. Is an inner product on the tangent space $T_x(M)$ we will frequently use thee alternative notation $(\cdot,\cdot)_x = g_x(\cdot,\cdot)$ the length of a tangent vector $v \in T_x(M)$ is defined as usual $|v|_x = g_x(v,v)^{1/2}$. If $v : [a,b] \to M$ is a piecewise smooth path, then we defined is length by $L(v) = \int_a^b |v(t)|\,dt$. If we choose local coordinates $(x^1, \ldots, x^n)$ on M ,then we get a local description of g as.

(37) $$\left[\, g = g_{ij}\left(dx^i, dx^j\right) \right], \left[\, g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) \right]$$

Proposition 2.8.4

Let be a smooth manifold, and denote by R_M the set of Riemannian metrics on M ,then R_M is a non –empty convex cone in the linear of symmetric tensor

Proof :

The only thing that is not obvious is that R_M is non-empty we will use again partitions of unity . Cover M by coordinate neighborhoods $(U_\alpha)_{\alpha \in A}$. Let x^j be a collection of local coordinates on U_α . Using these local coordinates we can construct by hand the metric g_α on U_α by $g_\alpha = \left(dx^1\right)_\alpha + \ldots + \left(dx^n\right)_\alpha$

now , pick a partition of unity $B \subset C_0^\infty(M)$ subordinated to cover U_α (i.e) there exists a map $\phi : B \to A$ such that $\forall \beta \in B \subset U_{\phi \in B}$ then define $g = \sum_{\beta \in B} \beta g\, \phi(B)$ The reader can check easily g is well defined ,and it is indeed a Riemann metric on M .

Example 2.8.5 [The Euclidean Space]

The space R^n has a natural Riemann metric $g_0 = dx^1, \ldots, dx^n$ The geometry of $\left(R^n, g_0\right)$ is the classical Euclidean geometry

Example 2.8.6 [Induced Metrics On Sub manifolds]

Let (M, g) be Riemann manifold and $S \subset M$ a sub manifold if $S \to M$, denotes the natural inclusion then we obtain by pull back a metric on S, $g^S = i\, g^* = g / S$,. For example , any invertible symmetric $n \times n$ matrix defines a quadratic hyper surface in R^n by $H_A = \{x \in R^n, (A_x, x) = 1\}$ where $[\cdot,\cdot]$ denotes the Euclidean inner on R^n, H_A has a natural .

Remark 2.8.7

On any manifold there exist many Riemannian metrics , and there is not natural way of selecting on of them . One can visualize a Riemannian structure as defining " shape " of the manifold . For example , the unit sphere $x^2 + y^2 + z^2 = 1$, is diffeomorphic to the ellipsoid $(x^2/1) + (y^2/2^2) + (z^2/3^3) = 1$,but they look "different" by However , appearances may be deceiving in is illustrated the deformation of a cylinder they look different ,but the metric structures are the same since we have not change length of curves on our sheep . the conclusion to be drawn from these two examples is that we have to be very careful when we use the attribute "different".

Example 2.8.8[The Hyperbolic Plane]

The Poincare model of the hyperbolic plane is the Riemannian manifold (D, g) where D is the unit open disk in the plan R^n and the metric g is given by .

$$\left[g = \frac{1}{1 - x^2 - y^2}\left(dx^2 + dy^2\right) \right] \tag{39}$$

Example 2.8.9

Left Invariant Metrics on lie groups Consider a lie group G ,and denote by L_G its lie algebra then any inner product $\langle \cdot,\cdot \rangle$ on L_G induces a Riemannian metric $h = \langle \cdot,\cdot \rangle_g$ on G defined by.

$$\begin{cases} h_g(x,y) = \langle x, y \rangle_g = \langle L_g^{-1} \rangle * X, (L_g^{-1}) * Y \\ \forall : g \in G, X \quad y \in T_g(G) \end{cases} \tag{40}$$

Where $(L_g^{-1})_* : T_g(G) \to T_1(G)$ is the differential at $g \in G$ of the left translation map L_g^{-1} . One checks easily that check easily that the correspondence $G \in g \to \langle \cdot,\cdot \rangle$ is a smooth tensor field, and it is left invariant (i,e) $L_g^* h = h \quad \forall g \in G$. If G is also compact, we can use the averaging technician to produce metrics which are both left and right invariant.

III. Conclusion

The paper study Riemannian differenterentiable manifolds is a generalization of locally Euclidean E^n in every point has a neighbored is called a chart homeomorphism, so that many concepts from as differentiability manifolds. We give the basic definitions, theorems and properties of Laplacian Riemannian manifolds becomes the spectrum of compact support M and Direct commutation of the spectrum, and spectral geometry of operators de Rahm.

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