International Journal of Engineering Science Invention
ISSN (Online): 2319 – 6734, ISSN (Print): 2319 – 6726
www.ijesi.org Volume 3 Issue 8 ‖ August 2014 ‖ PP.49-53

Establishment of New Special Deductions from Gauss Divergence Theorem in a Vector Field

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ABSTRACT : *Stokes proved that it is possible to express a surface integral in terms of a line integral round the boundary curve.Just as Green employed Stokes theorem in a vector field to establish another theorem (deduction), Gauss divergence theorem is employed herein to establish new special deductions i.e. equations (10), (12) and (13). The equations are tested with real live problems and the responses are positive.*

KEYWORDS: *Gauss divergence theorem, vector field, deduction.*

I. INTRODUCTION

It is known that the surface integral of $\bar{F}$ over S represents the lines of force diverging out of S. if the surface S encloses a volume V then, the quantity $\frac{\iint_S \bar{F}.dS}{V}$ gives the lines of force which diverge per unit volume and $\lim_{V\to 0}\left(\frac{\iint_S \bar{F}.dS}{V}\right)$represents the lines which diverge from the point to which the volume reduces in the limiting position. This is defined as the divergence of $\bar{F}$. Gauss theorem can be deduced as a consequence of this definition of divergence (Sastry, 1986; Sokolnikoff, 1941; Gupta, 2004).
Just as Green established this theorem from Stokes theorem, i.e.

$$\iint_s \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dxdy = \oint_c (Pdx + Qdy) \quad \text{……………………………………………(1)}$$

From $\int_s curl\ \bar{F}.dS = \iint_C \bar{F}.d\bar{r}$, ……………………………………………(2)

new special deductions are to be established herein from the Gauss divergence theorem i.e. that the volume integral (triple integral) on the left hand-side can be expressed as a surface integral (double integral) on the right-hand side (Lass, 1950; Philips, 1933),

$$\int_V div\,\bar{F}.dv = \int_S div\,\bar{F}.dS \quad \text{………………………………….(3)}$$

The objective is to establish these new special deductions from Gauss divergence theorem. Triple integrals taken over a region R in space can be transformed into surface integrals over the boundary surface S of R (and conversely) by the divergence theorem of Gauss. This is of practical interest because one of the two kinds of integral is often simpler than the other. It also helps in establishing fundamental equations in fluid flow, heat conduction, etc. (Marshal *et al.*, 1947; Dass, 1996; Sastry, 1986).

II. INTEGRAL THEORY

Most of the integrals we encounter in vector analysis are scalar quantities. For instance, given a vector function $\bar{F}$(x,y,z), we are often interested in the integral of its tangential component along a curve c or in the integral of its normal component over a surface S (Wylie, 1985). In the first case, if $\bar{R}$ is the vector from the origin to a general point of C so that$\frac{d\bar{R}}{ds} \equiv \bar{T}$ is the unit vector tangent to C at a general point, then $\bar{F}.\bar{T}$ is the tangential component of$\bar{F}$ and

$$\int_C \bar{F}.\bar{T}dS = \int_C \bar{F}.\frac{d\bar{R}}{ds}dS = \int_C \bar{F}.dR \quad \text{……………………………………………..(4)}$$

is the integral of this component along the curve C. In the second case, if n is the unit normal to S at a general point, then $\bar{F}.\bar{n}$ is the normal component of $\bar{F}$ and

$$\iint_s \bar{F}.\bar{n}\ dS \quad \text{………………………………….(5)}$$

is the integral of this component over the surface S. Other scalar integrals of frequent occurrence are the surface integral of the normal component of the curl of $\bar{F}$

$$\iint_s (\nabla x\bar{F}).\bar{n}\,dS \quad \text{…………………………………….….(6)}$$

and the volume integral of the divergence of $\bar{F}$

$\iiint_V \nabla.\bar{F}\, dV$ ………………………………….(7)

Fundamental in many of the applications of vector analysis is the so-called divergence theorem, which asserts the equality of the intervals (2) and (7) when V is the volume bounded by the closed regular surface S (Wylie, 1985; Hildebrand, 1947).

Gauss divergence theorem: If $\bar{F}$ (x,y,z) and $\nabla.\bar{F}$ are continuous over the closed regular surface S and its interior V, and if n is the unit vector perpendicular to S at a general point and extending outward from S, then:

$$\iint_S \bar{F}.\bar{n}\ dS = \iiint_V \nabla.\bar{F}\, dV \qquad \text{…………………………………………..(8)}$$

III. NEW SPECIAL DEDUCTIONS FROM GAUSS THEOREM

If F = Pi + Qj + Rk is a vector field,

Where P = P (x,y,z), Q = Q(x,y,z) and R = R(x,y,z), over the surface $a_x + a_y + a_z = C$.

The Gauss divergence theorem states that:

$$\int_V div\, \bar{F}.dV = \int_S \bar{F}.dS \qquad \text{……………………………….……….(3)}$$

For $\int_V div\, \bar{F}.dV$;

$$\text{div}\bar{F} = \nabla.\text{F} = \left[\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right].(Pi + Qj + Rk) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$\therefore \int_V div\, \bar{F}.dV = \iiint_V \left[\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right] dxdydz \qquad \text{…………………………………………..(9)}$$

Let's deal with $\bar{F}.dS, s: a_x + a_y + a_z - C = 0$

$$\text{Now } \nabla S = \frac{\partial S}{\partial x}i + \frac{\partial S}{\partial y}j + \frac{\partial S}{\partial z}k = a_x^1 i + a_y^1 j + a_z^1 k$$

$$\bar{n} = \frac{\nabla S}{|\nabla S|} = \frac{a_x^1 i + a_y^1 j + a_z^1 k}{\sqrt{(a_x^1)^2 + (a_y^1)^2 + (a_z^1)^2}}$$

$$\therefore d\bar{S} = \bar{n}\, dS = \frac{a_x^1 i + a_y^1 j + a_z^1}{\sqrt{(a_x^1)^2 + (a_y^1)^2 + (a_z^1)^2}} dS$$

$$\int_S F.dS = \int_S F.ndS = \int_S (Pi + Qj + Rk).\frac{a_x^1 i + a_y^1 j + a_z^1 k}{\sqrt{(a_x^1)^2 + (a_y^1)^2 + (a_z^1)^2}} dS \text{……………………………….(10)}$$

From (3), (8) and (10),

$$\iiint_V \left[\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right] dxdydz = \int_S (Pi + Qj + Rk).\frac{a_x^1 i + a_y^1 j + a_z^1 k}{\sqrt{(a_x^1)^2 + (a_y^1)^2 + (a_z^1)^2}} dS \text{……………………..(11)}$$

The surface area term dS depends on geometry of the figure in question.

If we project dS on to yz-plane, dR_x = dS (Cosα)

$$\text{Cos}\alpha = \bar{n}.i = \frac{a_x^1 i + a_y^1 j + a_z^1 k}{\sqrt{(a_x^1)^2 + (a_y^1)^2 + (a_z^1)^2}}.i = \frac{a_x^1}{\sqrt{(a_x^1)^2 + (a_y^1)^2 + (a_z^1)^2}}$$

If we project dS on to xz-plane, dR_y = dS (Cosβ)

$$\text{Cos}\beta = \bar{n}.j = \frac{a_x^1 i + a_y^1 j + a_z^1 k}{\sqrt{(a_x^1)^2 + (a_y^1)^2 + (a_z^1)^2}}.j = \frac{a_y^1}{\sqrt{(a_x^1)^2 + (a_y^1)^2 + (a_z^1)^2}}$$

If we project dS on to xy-plane, dR_z = dS (Cosϒ)

$$\text{Cos}\Upsilon = \bar{n}.k = \frac{a_x^1 i + a_y^1 j + a_z^1 k}{\sqrt{(a_x^1)^2 + (a_y^1)^2 + (a_z^1)^2}}.k = \frac{a_z^1}{\sqrt{(a_x^1)^2 + (a_y^1)^2 + (a_z^1)^2}}$$

Hence, dS = $dS_X + dS_Y + dS_Z$

$$= \frac{dR_X}{Cos\alpha} + \frac{dR_Y}{Cos\beta} + \frac{dR_Z}{Cos\Upsilon} = \frac{dydz}{\bar{n}.i} + \frac{dxdz}{\bar{n}.j} + \frac{dxdy}{\bar{n}.k}$$

$$= \sqrt{(a_x^1)^2 + (a_y^1)^2 + (a_z^1)^2}\left[\frac{dydz}{(a_x^1} + \frac{dxdz}{a_y^1} + \frac{dxdy}{a_z^1}\right] \text{……………………………………(12)}$$

Substituting (12) into (11) and simplifying, we have:

$$\iiint_V \left[\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right] dxdydz = \iint_S (Pa_x^1 + Qa_y^1 + Ra_z^1)\left[\frac{dydz}{(a_x^1} + \frac{dxdz}{a_y^1} + \frac{dxdy}{a_z^1}\right] \text{………… ……(13)}$$

For two opposite xy – planes, dS_z =(±k) dxdy

For two opposite yz – planes, dS_x = (±i) dydz

For two opposite xz – plane, $dS_y = (\pm j)\,dxdz$

$$\iiint_V \left[\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right] dxdydz$$

$$= \iint_s (Pi + Qj + Rk)\,[(\pm i)dydz + (\pm j)dxdz + (\pm k)\,dxdy\,] \quad \text{...(14)}$$

For cylinders, $(r, \emptyset, z)$, put $x = r\mathrm{Cos}\emptyset$, $y = r\,\mathrm{Sin}\emptyset$, $z = z$, and $dS = r\,d\emptyset\,dz$, $dv = r\,dr\,d\emptyset\,dz$

For spheres, $(r, \theta, \emptyset)$, Put $x = r\,\mathrm{Sin}\theta\mathrm{Cos}\emptyset$, $y = r\,\mathrm{Sin}\theta\mathrm{Sin}\emptyset$, $z = r\,\mathrm{Cos}\theta$, $dS = r^2\mathrm{Sin}\theta\,dr d\theta d\emptyset$

If S does not have the property of being exactly two-valued over its projections on each of the coordinate planes, then we can partition its interior V into sub-regionsV_i whose boundaries S_i do have this property. Now, applying our limited result to each of these regions, we obtain a set of equations of the form:

$$\iint_{si} \bar{F}.\bar{n}\; dS = \iiint_{Vi} \nabla.\bar{F}\, dV \quad \text{………………………………………….(15)}$$

If these are added, the sum of the volume integrals is, of course, just the integral of $\bar{\nabla}.\bar{F}$ throughout the entire volume V. the sum of the surface integrals is equal to the integral of $\bar{n}.\bar{F}$ over the original surface S plus a set of integrals over the auxiliary boundary surfaces which were introduces when V was subdivided. These cancel in pairs, however, since the integration extends twice over each interface, with integrands which are identical except for the oppositely directed unit normal they contain as factors (Wylie, 1985).

The theorem can be extended to surfaces which are such that lines parallel to the coordinate axes meet them in more than two points (Gupta, 2004).

Statement: If P = P (x,y,z), Q = Q (x,y,z), R = R (x,y,z), $\frac{\partial P}{\partial x}, \frac{\partial Q}{\partial y}, \frac{\partial R}{\partial z}$ can be continuous functions over a closed bounded region R in space whose boundary is a piecewise smooth orientable surface S, in x-y, y-z, and z-x planes, then equations (11), (13) and (14).

IV. RESULTS

Two examples curled from Stroud, K. (1994) to illustrate the theorem new special deductions are presented below:

Example 1: Verify the divergence theorem for the vector field: $F = x^2i + zj + jk$, taken over the region bounded by the planes: Z = 0, z = 2, x = 0, x = 1, y = 0, y = 3.

(a) To find $\int_V div\,\bar{F}.dV$

$$\mathrm{div}\bar{F} = \nabla.F = \left[\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right].(x^2i + zj + yk)$$

$$= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(y) = 2x + 0 + 0 = 2x$$

$$\int_V div\,\bar{F}.dv = \int_V 2xdV = \iiint_V 2xdxdydz$$

Inserting the limits and completing the integration,

For $\int_V div\,\bar{F}.dV = \int_0^1\int_0^3\int_0^2 2x\,dz\,dy\,dx = \int_0^1\int_0^3 [2xz]^2$

$= \int_0^1 [4xy]_0^3 dx = \int_0^1 12xdx = [6x^2]_0^1 = 6$

(b) To find $\int_S F.dS$ i.e. $\int_S F.\bar{n}dS$ i.e.

(i)S_1 (base): z = 0; $\bar{n}$ = -k (outwards and downwards)

$F = x^2i + yk$, $dS_1 = dxdy$

$$\int_{s_1} F\bar{n}.dS = \iint_{s_1}(x^2i + yk).(-k)dydx = \int_0^1\int_0^3(-y)dydx = \int_0^1\left[\frac{-y2}{2}\right]_0^3 dx = \frac{-9}{2}$$

(ii) S_2 (top): z = 2: n = k, $dS_2 = dxdy$

For $\int_{s_2} F\bar{n}.dS = \iint_{s_2}(x^2i + 2j + yk).(-k)dydx = \int_0^1\int_0^3 ydydx = \frac{9}{2}$

(iii) S_3 (right-hand end): y = 3; $\bar{n}$ = j, dS3 = dxdz

$F = x^2i + zj + yk$

$$\therefore \int_{s_3} F.\bar{n}dS = \iint_{s_3}(x^2i + 2j + 3k).(-j)dzdx = \int_0^1\int_0^2 zdzdx = \int_0^1\left[\frac{2^2}{2}\right]_0^2 dx = \int_0^1 2dx = 2$$

(iv) S_4 (left-hand end): y = 0; $\bar{n} = -j$, $dS_4 = dxdz$

For $\int_{s_4}\bar{F}.\bar{n}dS = \iint_{s_4}(x^2i + zj + yk).(-j)dzdx$

$$= \int_0^1 \int_0^2 (-z)dzdx = \int_0^1 \left[\frac{-z^2}{2}\right]_0^2 dx = \int_0^1 (-2)dx = -2$$

(v) S_5 (front): x = 1: $\bar{n} = 1$, dS_5 = dydz

$\int_{s_5} \bar{F}.\bar{n}dS = \iint_{s_5} (\text{i} + \text{zj} + \text{yk}).(-\text{i})\text{dydz} = \iint_{s_5} 1dydz = 6$

(vi) S_6 (back): x = 0: $\bar{n} = 1$, dS_6 = dydz

$\int_{s_6} \bar{F}.\bar{n}dS = \iint_{s_6} (\text{zj} + \text{yk}).(-\text{i})\text{dydz} = 0$

For the whole surface S, we therefore have:

$$\int_S F.dS = -\frac{9}{2} + \frac{9}{2} + 2 - 2 + 6 + 0 = 6$$

nd from our previous work in section (a) $\int_V div\, \bar{F}.dV = 6$

We have therefore verified as required that, in this example

$$\int_V div\, \bar{F}.dV = \int_S F.dS$$

Example 2: Verify the Gauss divergence theorem for the vector field: F = xi + 2j +2^2k, taken over the region bounded by the planes z= 0, z = 4, x = 0, y = 0 and the surface $x^2 + y^2 = 4$ in the first octant.

(a) div F = $\bar{\nabla}.F = \left[\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right]$.(xi + 2j + 2^2k)

:.$\int_V div\, \bar{F}.dV = \int_V \nabla.FdV = \iiint_V (1 + 2x)dxdydz$

Changing to cylindrical polar coordinates (p, Ø, z)

x = pCosØ; y = pSinØ; z = z; dV = p dp dØ dz

Transforming the variables and inserting the appropriate limits, we then have:

For $\int_V \text{div FdV} = \int_0^{\frac{\pi}{2}} \int_0^2 \int_0^4 (1 + 2z)\, dz\, pdp\, d\emptyset$ =

$$\int_0^{\frac{\pi}{2}} \int_0^2 [z + 22]_0^4\, pdpd\emptyset = \int_0^{\frac{\pi}{2}} \int_0^2 20\, pdpd\Phi = \int_0^{\frac{\pi}{2}} [10p2]_0^4\, d\Phi = \int_0^{\frac{\pi}{2}} 40\, d\Phi = 20\pi$$

(b) Now we evaluate $\int_S F.dS$ over the closed surface.

(i) S_1: z = 0 ; n = k ; F = xi + 2j

:.$\int_{S_1} \bar{F}.\bar{n}dS = \int_{S_1} (xi + 2j)(-k)dS = 0$

(ii) S_2: x = 4; n = k; F = xi + 2j + 16k

:. $\int_{S_2} \bar{F}.\bar{n}dS = \int_{S_2} (xi + 2j + 16k)(k)dS = \int_{S_2} 16\, dS = 16\left(\frac{4\pi}{4}\right) = 16\pi$

For we have

(iii) S_3: y = 0; n = -j; F = xi +2j + z^2k

:. $\int_{S_3} \bar{F}.\bar{n}dS = \int_{S_3} (xi + 2j + z^2k)\,(-j)\, dS = \int_{S_3} (-2)dS = -2\,(8) = -16$

(iv) S_4: x = 0; n = -j; F = 2j + z^2k

:. $\int_{S_4} \bar{F}.\bar{n}dS = \int_{S_4} (xi + 2j + z^2k)\,(-i)\, dS = 0$

Finally, we have:

(v) S_5: $x^2 + y^2 - 4 = 0$

$$n = \frac{\nabla S}{|\nabla S|} = \frac{2xi + 2yj}{\sqrt{4x^2 + 4y^2}} = \frac{xi + yj}{2}$$

$$\therefore \int_{S_5} \bar{F}.\bar{n}dS = \int_{S_5} (xi + 2j + z^2k).\left(\frac{xi + yj}{2}\right) dS = \frac{1}{2}\int_{S_5} (x^2 + 2y)\, dS$$

Converting to cylindrical polar coordinates, this gives

Since we have $\int_{S_5} \bar{F}.\bar{n}dS = \frac{1}{2}\int_{S_5} (x^2 + 2y)\, dS$

Also, $x = 2\, Cos\, \emptyset; y = 2\, Sin\, \emptyset; z = z; dS = 2\, d\emptyset\, dz$

$$\therefore \int_{S_5} \bar{F}.\bar{n}dS = \frac{1}{2}\int_0^4 \int_0^{\pi/2} (4Cos^2\emptyset + 4 Sin\,\emptyset)2d\emptyset\ dz$$

$$= 2\int_0^4 \int_0^{\pi/2} (1 + Cos\,2\emptyset + 2\,Sin\,\emptyset)2d\emptyset\ dz$$

$$= 2\int_0^4 \left[\left(\emptyset + \frac{Sin\,2\emptyset}{2}\right) - 2\,Cos\emptyset\right]_0^{\pi/2} dz = 2\int_0^4 \left(\frac{\pi}{2} + 2\right) dz = 4\pi + 16$$

Therefore, for the total surface S:

$$\int_S \bar{F}.\bar{n}dS = 0 + 16\pi - 16 + 0 + 4\pi + 16 = 20\pi$$

$$\therefore \int_V div\,\bar{F}.dV = \int_S F.dS = 20\pi$$

V. DISCUSSION

In the results illustrated, it will be remembered that, for a closed surface, the normal vectors at all points are drawn in an outward direction.

In the Gauss theorem, the vector point function $\bar{F}$ denotes the velocity vector of an incompressible fluid of unit density and S denotes any closed surface drawn in the space of the fluid, enclosing a volume.

Since the scalar product F.n represents the velocity-component at a point of the surface S in the direction of the outward drawn normal, therefore, $\bar{F}.\bar{n}dS$ expresses the amount of fluid flowing out in unit time through the element of surface dS. As such, the integral round the surface S, i.e. $\int_S \bar{F}.\bar{n}dS$ gives the amount of fluid flowing out of the surface S in unit time. But in order to maintain the continuity of the flow, the total amount of fluid flowing outwards must be continually supplied so that inside the region there are sources producing fluid.

Now the $div\,\bar{F}$ at any point represents the amount of fluid passing through that point per unit time per unit volume. So, $div\,\bar{F}$ may be regarded as the source-intensity of the incompressible fluid at any point (Gupta, 2004).

VI. CONCLUSION

The total volume per second of a moving fluid flowing out from a closed surface S is equal to the total volume per second of fluid flowing out from all volume elements in S.

One of the most important uses of vector analysis is in the concise formulation of physical laws and the deviation of other results from those laws. For instance, the development of the concept of potential and obtaining the partial differential equation satisfied by the gravitational potential is one of the examples of this sort.

In most recent problems, the masses or changes which produce $\bar{F}$ are given, and it is required to find $\bar{F}$ itself.

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