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Some Results Concerning Nevanlinna Defects of The Euler's Gamma Function.

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ABSTRCT: *In this paper we have extended some basic results of Nevanlinna theory to Euler's gamma function which is known to be a meromorphic function.*

KEY WORDS*: Nevanlinna theory, Euler's gamma function,Nevanlinna defects.*

Preliminaries:

By a meromorphic function we shall always mean a transcendental meromorphic function in the plane. If f is a meromorphic function, $a \in \overline{C}$ and r > 0, we use the following notations of frequent use in Nevanlinna theory with their usual meaning: $m(r,a,f) = m\left(r \frac{1}{f-a}\right)$,

$n(r,a,f) = n\left(r, \frac{1}{f-a}\right), \bar{n}(r,a,f)$, N(r, a, f), $\overline{N}$(r, a, f), T(r, f) ,

$\delta(a,f), \Delta(a,f), \Theta(a,f)$ etc..as in [2]

As usual, if a = ∞, then by a zero of f-a, we mean a pole of f.

Introduction: The Euler's Gamma function $\Gamma(z)$ is given by

$$\Gamma(z) = z^{-1} e^{-\gamma z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right)^{-1} e^{\frac{z}{k}} \qquad (1)$$

Where $\gamma = \lim_{n\to\infty} \sum_{k=1}^{n} (k^{-1} - \log n)$ is Euler's constant.

Clearly $\Gamma(z)$ is meromorphic function with simple poles at $\{-k\}_{k=0}^{+\infty}$ and $\Gamma(z) \neq 0$ for z $\in$ C.

In [3], Zhuan Ye has proved the following.

Theorem A: with usual notations,

(1) $T(\Gamma, r) == (1 + o((1))\left(\frac{r}{\pi}\right) \log r$

(2) $\delta(\Gamma, o) = \delta(\Gamma, \infty) = 1,\ \delta(\Gamma, a) = 0$ for a ≠ 0, ∞ .

We wish to obtain some other results of Nevanlinna theory related to Euler's Gamma function and prove the following result.

Theorem 1 : Suppose $\Gamma(z)$ is the Euler Gamma – function as defined in (1) and let $\Gamma_i(z)$ (i=1, 2, …… p) be p (2 ≤ p ≤ ∞) distinct small meromorphic functions of finite order μ_{r_i} satisfying T(r, Γ_i) = o {T (r, Γ)} (r → ∞). If p = + ∞ , then for any ∈ >0 , there exists a positive integer q such that $K(L_q(\Gamma))$ < 1 - $\sum_{p=1}^{\infty} \delta(\Gamma_i, \Gamma)$, Where $L_q\,(\Gamma_q(z)) \quad = \dfrac{(-1)^p W(\Gamma,\Gamma_1,\Gamma_2....\Gamma_p)}{W(\Gamma_1,\Gamma_2....\Gamma_p)}$

$$= \Gamma^{(p)} + a_{p-1}\Gamma^{(p-1)} + a_{p-2}\Gamma^{(p-2)} + \ldots + a_0\Gamma$$

We require the following Lemma in our proof.

Lemma [1] Let the differential equation $w^{(k)} + a_{k-1}\, w^{(k=1)} + \ldots + a_0\, w = 0$ be satisfied in the complex plane by linearly independent meromorphic functions $f_1, f_2 \ldots f_k$.

Then the co-efficients $a_j(j=0, 1, \ldots, k-1)$ are meromorphic in the plane with the property

$$m(r, a_j) = O\left\{\log\left(\max_{i=1,\ldots k} T(r, f_i)\right)\right\}$$

Proof of Theorem :

First we, consider the $p = +\infty$.

Let $F(z) = \displaystyle\sum_{i=1}^{p} \frac{1}{\Gamma(z)-\Gamma_i(z)}$

Then, by an earlier result we know that,

$m(r, F) \geq \displaystyle\sum_{i=1}^{q} m\left(r, \frac{1}{\Gamma-\Gamma_i}\right) - o\{T, r, \Gamma)$ for any positive integer $q < \infty$.

Now, $\quad N(r, a_i) = \displaystyle\sum_{i=1}^{p} N(r, \Gamma_i^{(p)})$

$$\leq (p+1) \sum_{i=1}^{p} N(r, \Gamma_i)$$

$$= o\{T(r, \Gamma)\}$$

and $m(r, a_i) = o\{T(r, \Gamma)\}$, by the above Lemma.

Also, We have

$$\sum_{i=1}^{p} \frac{1}{\Gamma-\Gamma_i} = \frac{1}{L_q(\Gamma)} \sum_{i=1}^{q} \frac{L_q(\Gamma)}{\Gamma-\Gamma_i}$$

$$= \sum_{i=1}^{p} \frac{1}{L_q(\Gamma)} \sum_{i=1}^{q} \frac{L_q(\Gamma-\Gamma_i)}{\Gamma-\Gamma_i}$$

and $m\left(r, \displaystyle\sum_{i=1}^{q} \frac{L_q(\Gamma-\Gamma_i)}{\Gamma-\Gamma_i}\right) \leq \displaystyle\sum_{i=1}^{q} \sum_{j=1}^{p} m\left(r, \frac{(\Gamma-\Gamma_i^{(j)})}{\Gamma-\Gamma_i}\right)$

$$+ \sum_{t=0}^{p} m(r, a_t) + O(1).$$

Hence, We have $m\left(r, \displaystyle\sum_{i=1}^{q} \frac{1}{\Gamma-\Gamma_i}\right) \leq m\left(r, \frac{1}{L_q(\Gamma)}\right) + o\{T(r, \Gamma)\}$

Therefore, $\quad \displaystyle\sum_{i=1}^{q} m\left(r, \frac{1}{\Gamma-\Gamma_i}\right) \leq m\left(r, \frac{1}{L_q(\Gamma)}\right) + o\{T(r, \Gamma)\}$

Thus, we obtain

$$\sum_{i=1}^{q} \delta(\Gamma_i, \Gamma) \leq \lim_{r\to\infty} \sum_{i=1}^{q} \frac{m\left(r, \dfrac{1}{\Gamma-\Gamma_i}\right)}{T(r, \Gamma)}$$

$$\leq \underline{\lim}_{r\to\infty} \frac{m\left(r, \frac{1}{L_q(\Gamma)}\right)}{T(r,\Gamma)}$$

on the other hand,

$$1 - K(L_q(\Gamma)) = \overline{\lim}_{r\to\infty} \frac{N(r, L_q\Gamma) + N\left(r, \frac{1}{L_q(\Gamma)}\right)}{T(r, \Gamma_q(\Gamma))}$$

$$= \overline{\lim}_{r\to\infty} \frac{m\left(r, \frac{1}{L_q(\Gamma)}\right) - O(1) - N(r, L_q(\Gamma))}{T(r, \Gamma_q(\Gamma))}$$

using the First Fundamental Theorem.

Hence, $1 - K(L_q, (\Gamma)) \geq \underline{\lim}_{r\to\infty} \frac{m\left(r, \frac{1}{L_q(\Gamma)}\right) - O\{T(r,\Gamma)\}}{T(r, L_q(\Gamma))}$

$$\geq \underline{\lim}_{r\to\infty} \frac{m\left(r, \frac{1}{L_q(\Gamma)}\right) - O\{T(r,\Gamma)\}}{T(r,\Gamma)}$$

$$= \underline{\lim}_{r\to\infty} \frac{m\left(r, \frac{1}{L_q(\Gamma)}\right)}{T(r,\Gamma)}$$

Further, for any $\in 0$, there exists a positive integer.

q_0 $(0 < q_0 < +\infty)$ and $\{\Gamma_{it}\}_{t=1}^{q_0} \subset \{\Gamma_i\}_{i=1}^{\infty}$

Such that

$$\sum_{i=1}^{\infty} \delta(\Gamma_i, \Gamma) < \sum_{t=1}^{q_0} \delta(\Gamma_{it}, \Gamma).$$

Thus, we have, $\sum_{i=1}^{\infty} \delta(\Gamma_i, \Gamma) - \varepsilon < \sum_{t=1}^{q_0} \delta(\Gamma_{it}, \Gamma)$

$$\leq 1 - K(L_q, (\Gamma)).$$

Hence, we have $K(L_{q0}(\Gamma)) < 1 - \sum_{i=1}^{\infty} \delta(\Gamma_i, \Gamma) + \varepsilon$ for a positive integer q_0.

If p is finite, then in the above discussion we may take $q = q_0 = p$.
This proves the result.

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