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Exceptional Values of Meromorphic Functions and Differential Polynomials

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ABSTRACT: *Let f (z) be a transcendental meromorphic function of finite order with four distinct evP for simple zeros and k be a positive integer. We wish to improve theresult of Hong xun Yi by introducing the notion of the order of multiplicity for the zeros of f (z).*

I. INTRODUCTION

We call a an evP (exceptional value in the sense of Picard) for f if n(r, a, f) = O (1). Thus, a is an evP for f if f-a has only a finite number of zeros.
In [9], Singh has proved the following.

Theorem A Let f(z) be a transcendental meromorphic function of finite order with four (finite or infinite) distinct evP for simple zeros. Then,

$$\lim_{r\to\infty} \frac{T(r, f^{1})}{T(r, f)} = \frac{3}{2}$$

Later, Hong Xun Yi observed the following in [6]
Theorem B Let f(z) be a transcendental meromorphic function of finite order with four distinct evP for simple zeros.
(i) If ∞ is an evP for simple zeros of f(z), then

$$\lim_{r\to\infty} \frac{T(r, f')}{T(r, f)} = \frac{3}{2} \qquad \text{and}$$

(ii) if ∞ is not an evP for simple zeros of f(z), then.

$$\lim_{r\to\infty} \frac{T(r, f')}{T(r, f)} = 2$$

Also, Yi has given the generalization to Theorem B as follows.
Theorem C Let f(z) be a transcendental meromorphic function of finite order with four distinct evP for simple zeros and k be a positive integer.
(i) If ∞ is an evP for simple zeros of f(z),then

$$\lim_{r\to\infty} \frac{T(r, f^{(k)})}{T(r, f)} = \frac{1}{2}k + 1$$

and (ii) if ∞ is not an evP for simple zeros of f(z), Then,

$$\lim_{r\to\infty} \frac{T(r, f^{(k)})}{T(r, f)} = k + 1$$

We wish to improve this result by introducing the notion of the order of multiplicity for the zeros of f(z).
In order to establish our main result, first we introduce the following notation.
Definition Let f(z) be a transcendental meromorphic function and $a \in \overline{C}$.
We denote by $n_p(r, a, f)$ the number of zeros of f(z)-a in $|z| \le r$, where a zero of multiplicity ≤ p is counted according to its multiplicity and a zero of multiplicity > p is counted exactly p times.

$N_p(r,a,f)$ is defined in terms of $n_p(r,a,f)$ in the usual way.

We define $\delta_p(a,f)=1-\overline{\lim_{r\to\infty}}\dfrac{N_p(r,a,f)}{T(r,f)}$

Our main result is the following.

Theorem 1 Let f(z) be a transcendental meromorphic function of finite order and k be a positive integer.

If $\sum_{a\in\overline{C}}\delta_p(a,f)=4$, then (1)

$$\lim_{r\to\infty}\frac{T(r,f^{(k)})}{T(r,f)}=k+1-\frac{pk}{p+1}\ \delta_p(\infty,f)$$

Proof Let $a_1, a_2,\ldots, a_q$ be distinct complex numbers.

By the Second Fundamental Theorem, we have

$$(q-1)T(r,f)<\sum_{i=1}^{q}\overline{N}(r,a_i,f)+\overline{N}(r,f)+O\{\log r\} \tag{2}$$

Again, $\overline{N}(r,a_i,f)\le\dfrac{p}{p+1}N_p(r,a_i,f)+\dfrac{1}{p+1}\ N_p(r,a_i,f)$ (3)

$$\le\frac{p}{p+1}N_p(r,a_i,f)+\frac{1}{p+1}\ T(r,f)+O(1).$$

From (2) and (3), we have

$$(q-1)T(r,f)<\frac{p}{p+1}\sum_{i=1}^{q}\ N_p(r,a_i,f)+\frac{1}{p+1}\ qT(r,f)+\overline{N}(r,f)+O\{\log r\} \tag{4}$$

Thus, $\underline{\lim}_{r\to\infty}\dfrac{\overline{N}(r,f)}{T(r,f)}\ge\dfrac{p}{p+1}\sum_{i=1}^{q}\ \delta_p(a_i,f)-1,$ (5)

after simplification.

Letting $q\to\infty$, we get

$$\underline{\lim}_{r\to\infty}\frac{\overline{N}(r,f)}{T(r,f)}\ge\frac{p}{p+1}\sum_{a\in\overline{C}}\ \delta_p(a,f)-1$$

$$=1-\frac{p}{p+1}\ \delta_p(\infty,f) \tag{6}$$

On the other hand,

$$\overline{N}(r,f)\le\frac{p}{p+1}N_p(r,f)+\frac{1}{p+1}\ N_p(r,f) \tag{7}$$

Therefore, $\overline{N}(r,f)\le\dfrac{p}{p+1}N_p(r,f)+\dfrac{1}{p+1}\ T(r,f)$ (8)

Thus, $\overline{\lim_{r\to\infty}}\dfrac{\overline{N}(r,f)}{T(r,f)}\le\dfrac{p}{p+1}\overline{\lim_{r\to\infty}}\dfrac{N_p(r,f)}{T(r,f)}+\dfrac{1}{p+1}$

$$=1-\frac{p}{p+1}\ \delta_p(\infty,f)\text{ after simplification.} \tag{9}$$

From (6) and (9), we have,

$$\lim_{r\to\infty}\frac{\overline{N}(r,f)}{T(r,f)}=1-\frac{p}{p+1}\ \delta_p(\infty,f) \tag{10}$$

From (7) and (10), we have

$$\overline{\lim_{r\to\infty}} \frac{\overline{N}(r,f)}{T(r,f)} = 1 \quad \text{using the fact that } N(r,f) \le T(r,f) \tag{11}$$

Now, $T\left(r,f^{(k)}\right) = m\left(r,f^{(k)}\right) + N\left(r,f^{(k)}\right)$

$$= [m(r,f) + O(\log r)] + N(r,f) + k\overline{N}(r,f)$$

$$< T(r,f) + k\overline{N}(r,f) + O(\log r) \tag{12}$$

From (10), (11) and (12), we have

$$\lim_{r\to\infty} \frac{T\left(r,f^{(K)}\right)}{T(r,f)} = 1 + k\left[1 - \frac{p}{p+1} \quad \delta_p(\infty,f)\right]$$

Therefore, $\lim_{r\to\infty} \dfrac{T\left(r,f^{(k)}\right)}{T(r,f)} = (k+1)1 - \dfrac{pk}{p+1} \quad \delta_p(\infty,f)$ and hence the result.

Remarks 1 For simple zeros of f(z), we have $\delta_p(\infty,f) = \delta_1(\infty,f)$

Hence the above Theorem becomes

$$\lim_{r\to\infty} \frac{T(r,f^{(k)})}{T(r,f)} = (k+1) - \frac{1}{2}k \ \delta_1(\infty,f)$$

Which is the result of Hong-Xun-Yi

Remark 2 In particular, for simple zeros of f(z)

(a) If ∞ is an evp, then $\delta_1(\infty,f) = 1$

Therefore, $\lim_{r\to\infty} \dfrac{T\left(r,f^{(k)}\right)}{T(r,f)} = \dfrac{1}{2}k + 1$

and (b) if ∞ is not an evp, then $\delta_1(\infty,f) = 0$

Therefore, $\lim_{r\to\infty} \dfrac{T\left(r,f^{(k)}\right)}{T(r,f)} = k + 1$

which is Theorem C.

One can easily see that Theorem B follows by Theorem C by putting k = 1.

Now, we wish to extend Theorem 1 to differential polynomials.

Theorem 2 Let P[f] be a homogeneous differential polynomial in f having degree γ_p and weight Γ_p with $\sum_{a\neq\infty} \Theta\ (a,f) = 2.$

Then, $\lim_{r\to\infty} \dfrac{T(r,P[f])}{T(r,f)} = \Gamma_p$

To prove the above Theorem, we require the following Lemmas.

Lemma 1 If P[f] is a homogeneous differential polynomial in f, then

$$m(r,P[f]) \le \gamma_p m(r,f) + S(r,f)$$

Proof $m(r,P[f]) = m\left(r, \dfrac{P[f]}{f^{\gamma_p}}.f^{\gamma_p}\right)$

$$\le m\left(r, \frac{P[f]}{f^{\gamma_p}}\right) + m\left(r,f^{\gamma_p}\right) + S(r,f)]$$

$\le \gamma_p m(r,f) + S(r,f)$, by Milloux's Theorem.

Lemma 2 [7] If P[f] is a homogeneous differential polynomial in f, then

$$N(r,P[f]) \le \gamma_p N(r,f) + (\Gamma_p - \gamma_p)\ \overline{N}(r,f) + S(r,f)$$

Lemma 3 [7] If f is a meromorphic function with finite order such that $\sum_{a\neq\infty}\Theta(a,f)=2$, then

$$\lim_{r\to\infty}\frac{\overline{N}(r,f)}{T(r,f)}=1 \quad .$$

Lemma 4 [7] Suppose Q[f] is a differential polynomial in f. Let z_0 be a pole of f of order m and not a zero or a pole of the co-efficient of Q[f]. Then z_0 is a pole of Q[f] of order at most $m\gamma_Q+(\Gamma_Q-\gamma_Q)$

Proof of Theorem 2

Now, $T(r,P[f])=m(r,P[f])+N(r,P[f])$

Therefore, by Lemma 1 and Lemma 2, we have

$T(r,P[f])\leq\gamma_p\,T(r,f)+(\Gamma_p-\gamma_p)\,\overline{N}(r,f)+S(r,f).$

Hence, using Lemma 3, we get

$$\lim_{r\to\infty}\frac{T(r,P[f])}{T(r,f)}\leq\gamma_P+\Gamma_P-\gamma_P$$

which implies $\lim_{r\to\infty}\frac{T(r,P[f])}{T(r,f)}\leq\Gamma_p$

On the other hand, by Lemma 4, we have

$$N(r,P[f])=\text{Max}\{m\gamma_p+(\Gamma_p-\gamma_p)\}N(r,f)$$

Therefore, $N(r,P[f])\geq\text{Min}\{m\gamma_p+(\Gamma_p-\gamma_p)\}N(r,f)$

Therefore, $N(r,P[f])\geq\gamma_p N(r,f)+(\Gamma_p-\gamma_p)\ \overline{N}(r,f)$

Thus, $T(r,P[f])=m(r,P[f])+N(r,P[f])$

Therefore, $T(r,P[f])\geq\gamma_p m(r,f)+\gamma_p N(r,f)+(\Gamma_p-\gamma_p)\ \overline{N}(r,f)+S(r,f)$

Therefore, $T(r,P[f])\geq\gamma_p T(r,f)+(\Gamma_p-\gamma_p)\ \overline{N}(r,f)+S(r,f)$.

Hence $\lim_{r\to\infty}\frac{T(r,P[f])}{T(r,f)}\leq\Gamma_p$, in view of Lemma 3. (14)

From (13) and (14), we c-+**************

onclude that $\lim_{r\to\infty}\frac{T(r,P[f])}{T(r,f)}=\Gamma_p$

Hence the result.

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