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(*K, D*)-Even Edge-Graceful Labeling Of Odd Order Trees

Dr. S. Kousalya Devi
Associate Professor
PG and Research Department of Mathematics, Periyar EVR College, Trichy – 23, India.

ABSTRACT : *In this paper, we introduce the concept of (k, d)-even edge-graceful labeling and investigate (k, d)-even edge-graceful labeling of some odd order trees.*

KEYWORDS: *k-even edge-graceful labeling, k-even edge-graceful graph,* **(***k, d)-even edge-graceful labeling, (k, d)-even edge-graceful graph.*
AMS (MOS) Subject Classification (2010) : *05C78.*

I. INTRODUCTION

All graphs in this paper are finite, simple and undirected. Terms not defined here are used in the sense of Harary [7]. The symbols *V(G)* and *E(G)* will denote the vertex set and edge set of a graph G. Labeled graphs serve as useful models for a broad range of applications[1-4].Many studies in graph labeling refer to Rosa's research in 1967 [9]. A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. If the domain of the mapping is the set of vertices (or edges) then the labeling is called a vertex labeling (or an edge labeling). Graph labeling was first introduced in the late 1960's.In 1985, Lo [8] introduced the notion of edge-graceful graphs. In [5], Gayathri *et al.*, introduced the even-edge-graceful graphs. The concept of *k*-edge-graceful labeling was introduced in [10]. We have introduced a labeling called *k*-even edge-graceful labeling in [6]. In this paper, we introduce the concept of **(***k, d*)-even edge-graceful labeling and investigate the (*k, d*)-even edge-graceful labeling of some odd order trees.Throughout this paper, *k* and *d* denote any positive integer ≥ 1.

II. MAIN RESULTS

Definition 2.1

(*k, d*)-even edge-graceful labeling ((*k, d*)-*EEGL*) of a (p, q) graph $G\ (V, E)$ is an injection $f : E \rightarrow \{2k - 1, 2k - 1 + d, 2k - 1 + 2d, \ldots, 2k - 1 + (2q - 1)d\}$ such that the induced mapping f^+ defined on V by $f^+(x) = (\Sigma f(xy)) \pmod{2s}$ taken over all edges xy are distinct and even, where $s = max\ \{p, q\}$ and k, d are positive integers. That is $f^+(V) = \{0, 2, 4, \ldots, 2s–2\}$. A graph G that admits (*k, d*)-even edge-graceful labeling is called a (*k, d*)-even edge-graceful graph ((*k, d*)-*EEGG*).

Example 2.2

A (3, 5)-*EEGL* of $< K_{1,5} : K_{1,3} >$ is given in Figure 1.Here $p = 11$; $q = 10$;
$s = max\ \{p, q\} = 11$.

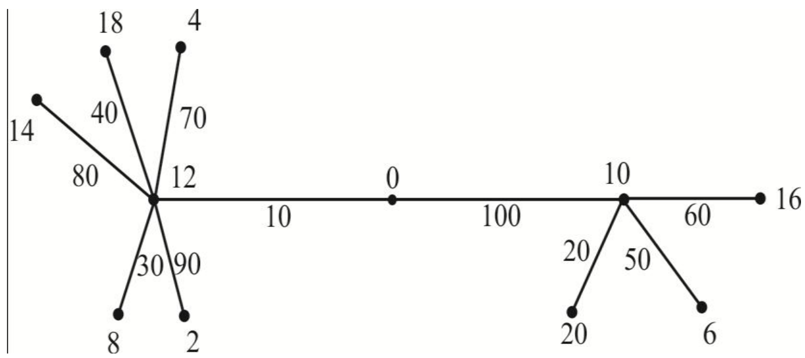


Figure 1 : (3, 5)-labeling of $< K_{1,5} : K_{1,3} >$

Remark 2.3

(i) (*k*, 1)-even edge-graceful labeling is a *k*-even edge-graceful labeling.
(ii) (1, 1)-even edge-graceful labeling is an even edge-graceful labeling.

Theorem 2.4

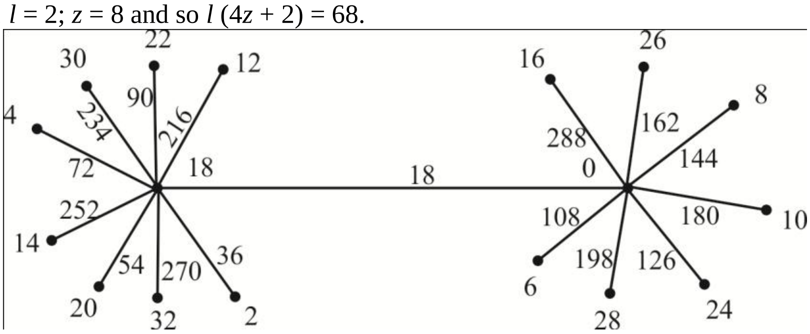
If a tree T on p vertices where p is odd has a (k, d)-*EEGL*, then T also has a $(lp+k, d)$-*EEGL* for any $l, k \geq 1$ and d is odd.

Proof

Given that p is odd. Therefore take $p = 2z + 1$ and so $q = 2z$ where z is a positive integer.
Let $f : E \to \{2k – 1, 2k – 1 + d, …, 2k – 1 + (4z – 1) d\}$ be a (k, d)-*EEGL* of the tree T. Then the induced vertex labels are $\{0, 2, 4, …., 4z\}$.
Now define a labeling $g : E \to \{2(l (2z + 1)+k) – 1, 2(l (2z + 1) + k) –1 + d,$
$…, 2(l (2z + 1) + k) – 1 + (4z – 1) d\}$ by
$g(e) = f(e) + l (4z + 2)$ for any edge $e \in T$.
Therefore for any vertex x in T,
$g^+(x) \quad = \Sigma g (xy) \pmod{4z + 2}$
$= \Sigma f(xy) \pmod{4z+2}$
$= f^+(x)$
Therefore, the induced vertex labels are again $\{0, 2, 4, …., 4z\}$.
Hence T has a $(lp + k, d)$ - EEGL for any $l, k \geq 1$ and d is odd.

Illustration 2.5

A (5, 9)-*EEGL* and its corresponding (39, 9)-*EEGL* of $B_{8,7}$ are given in Figure 2. Here $p = 17$; $q = 16$; $s = max \{p, q\} = 17$; $l = 2$; $z = 8$ and so $l (4z + 2) = 68$.



(5, 9)-*EEGL* of $B_{8,7}$

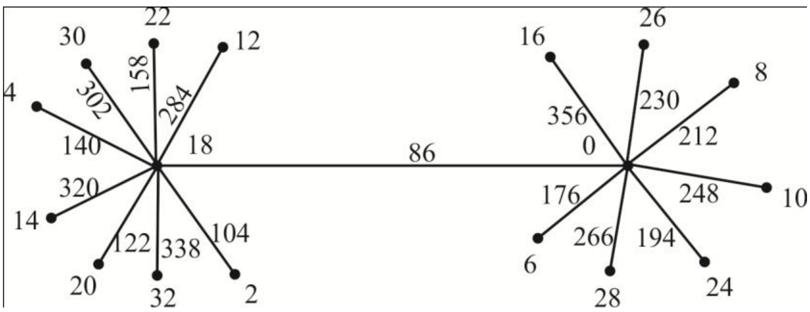


Figure 2 : (39,9)-*EEGL* of $B_{8,7}$

Corollary 2.6

If there is no (k,d)-*EEGL* for a tree T of odd order p, then there is no $(lp+k, d)$- *EEGL* for T where l, k and $d \geq 1$.

Proof

Given that p is odd. Therefore take $p = 2z + 1$ and so $q = 2z$ where z is a positive integer.
Let $f : E \to \{2 (l (2z + 1) + k) – 1, 2 (l (2z +1) + k) – 1 + d, ….,$
$2(l (2z + 1) + k) – 1) + (4z – 1) d\}$ be a $(lp + k, d)$-*EEGL* of the tree T. Then the induced vertex labels are $\{0, 2, 4, …, 4z\}$.
Now define a labeling
$g : E \to \{2k – 1, 2k – 1 + d, …, 2k – 1 + (4z – 1) d\}$ by
$g(e) = f(e) – l (4z + 2)$ for any edge $e \in T$.
Therefore for any vertex x in T,

$g^+(x) \;=\; \Sigma\, g(xy) \pmod{4z + 2}$
$= \;\Sigma\, f(xy) \pmod{4z + 2}$
$= \; f^+(x)$

Therefore the induced vertex labels are again $\{0, 2, 4, \ldots, 4z\}$.
Hence T has a (k, d)-*EEGL*.
This is a contradiction to our assumption that T has no (k, d)-*EEGL*.
Therefore there is no $(lp+k, d)$-*EEGL* for T where l, k and $d \geq 1$.

Illustration 2.7

$T(11,10)$is a tree with $p = 11$ and $q = 10$.

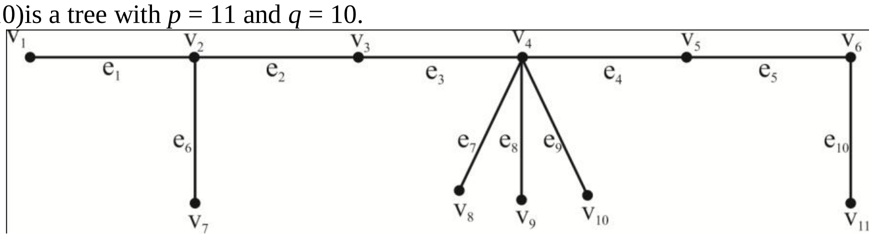


Figure 3 : T has no $(11l + 6, \mathrm{d})$-*EEGL* where $l \geq 0$ and $d \geq 1$.

Theorem 2.8

If atree T on p vertices where p is odd has a (k, d)-*EEGL*, then T also has a $(k, 2lp + d)$-*EEGL* for any l, $k \geq 1$ and d is odd.

Proof

Given that p is odd. Therefore take $p = 2z + 1$ and so $q = 2z$ where z is a positive integer.

Let $f : E \rightarrow \{2k - 1, 2k - 1 + d, \ldots., 2k - 1 + (4z - 1)\, d\}$ be a (k, d)-*EEGL* of the tree T. Then the induced vertex labels are $\{0, 2, 4, \ldots, 4z\}$.

Now define a labeling

$g: E \rightarrow \{2k - 1, 2k - 1 + 2l\,(2z + 1) + d, 2k - 1 + 2\,(2l\,(2z + 1) + d), \ldots,$
$2k - 1 + (4z - 1)\,(2l\,(2z + 1) + d)\}$ by

$g(e) = f(e) + li\,(4z + 2)$ if $f(e) = 2k - 1 + id$ for any edge $e \in T$ and
$i = 0,1,2,3,\ldots,(4z - 1)$.

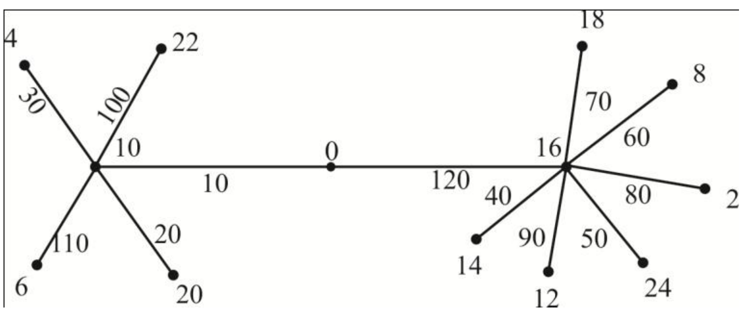
Therefore, for any vertex x in T,

$g^+(x) \;=\; \Sigma g\,(xy) \pmod{4z + 2}$
$= \;\Sigma f\,(xy) \pmod{4z + 2}$
$= \; f^+(x)$

Therefore the induced vertex labels are again $\{0, 2, 4, \ldots, 4z\}$.
Hence T has a $(k, 2lp + d)$-*EEGL* for any l, $k \geq 1$ and d is odd.

Illustration 2.9

A $(3, 5)$-*EEGL* and its corresponding $(3, 31)$-*EEGL* of $< K_{1,4} : K_{1,6} >$ are given in Figure 4. Here $p = 13$; $q = 12$; $s = max\,\{p, q\} = 13$; $l = 1$; $z = 6$ and so $l\,(4z + 2) = 26$.



(3, 5)-*EEGL* of $< K_{1,4} : K_{1,6} >$

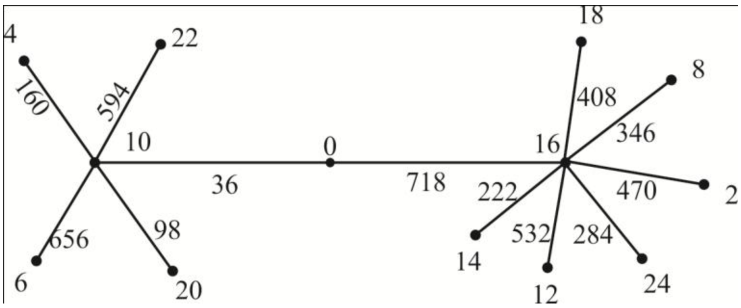


Figure 4 : (3, 31)-*EEGL* of < $K_{1,4}$: $K_{1,6}$ >

Corollary 2.10

If there is no (k, d)-*EEGL* for a tree T of odd order p, then there is no (k, $2lp + d$) - *EEGL* for T where l, k and $d \geq 1$.

Proof

Given that p is odd. Therefore take $p = 2z + 1$ and so $q = 2z$ where z is a positive integer.

Let $f : E \to \{2k - 1, 2k - 1 + 2l(2z + 1) + d, 2k - 1 + 2\,(2l\,(2z + 1) + d), \ldots,$
$2k-1 + (4z - 1)\,(2l\,(2z + 1) + d)\}$ be a
(k, $2lp + d$)-*EEGL* of the tree T.

Then the induced vertex labels are $\{0, 2, 4, \ldots, 4z\}$.

Now define a labeling

$g: E \to \{2k - 1, 2k - 1 + \mathrm{d} \ldots, 2k - 1 + (4z - 1)d\}$ by

$g(e) = f(e) - li(4z + 2)$ if $f(e) = 2k - 1 + id$ for any edge $e \in T$ and
$i = 0,1,2,3,\ldots, (4z - 1)$.

Therefore, for any vertex x in T,

$$\begin{aligned} g^+(x) &= \Sigma g\,(xy) (\mathrm{mod}\ 4z + 2) \\ &= \Sigma f\,(xy) (\mathrm{mod}\ 4z + 2) \\ &= f^+(x) \end{aligned}$$

Therefore the induced vertex labels are $\{0, 2, 4, \ldots, 4z\}$.

Hence T has a (k, d)-*EEGL*. This is a controduction to our assumption that T has no (k, d)-*EEGL*. Therefore there is no (k, $2lp + d$)-*EEGL* for T where l, k and $d \geq 1$.

Illustration 2.11

T (17, 16) is a tree with p =17 and q =16.

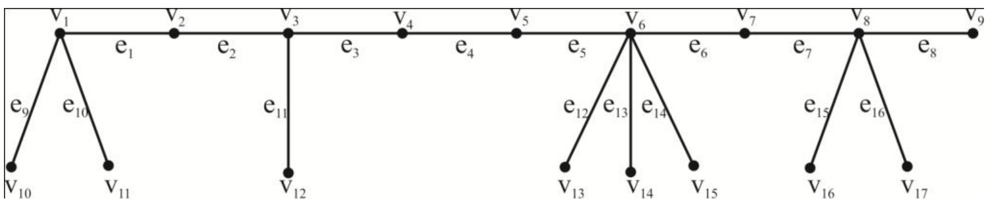


Figure 5 : *T* has no (9, 34*l* + *d*)-*EEGL* where $l \geq 0$ and $d \geq 1$.

Observation 2.12

1. Let f be a (k, d)-*EEGL* of odd order tree then d cannot be even and when d is odd,
   $f(E) = \{2k - 1 + d, 2k - 1 + 3d, 2k - 1 + 5d, \ldots, 2k - 1 + (2q - 1)d\}$
2. $\sum_{v \in V} f^+(v)\ (\mathrm{mod}\ 2q + 2) = 2\sum_{e \in E} f(e) (\mathrm{mod}\ 2q + 2)$ for any (k, d)-*EEGL* of a tree.

Lemma 2.13

If there is *a* (k, d)-*EEGL* for any tree T, then

$4qk - 3q + q^2\,(2d - 1) \equiv 0\ (\mathrm{mod}\ 2q + 2)$

Proof

Given T is a tree and T has a (k, d)-*EEGL*. Therefore we take

$f(E) = \{2k - 1 + d, 2k - 1 + 3d, 2k - 1 + 5d, \ldots, 2k - 1 + (2q - 1)\,d\}$.

Now by observation 2.12 (2),

$$\sum_{v\in V} f^{+}(v) \pmod{2q+2} = 2\sum_{e\in E} f(e) \pmod{2q+2}$$

Therefore

R.H.S. $= 2\,[(2k-1)+d+(2k-1)+3d+\ldots+(2k-1)+(2q-1)d] \pmod{2q+2}$

$= 2\,[q(2k-1)+d\,(1+3+5+\ldots.(2q-1))] \pmod{2q+2}$

$= 2\,[2qk-q+dq^2] \pmod{2q+2}$ …… (1)

And L.H.S. $= (0+2+4+\ldots.+2q) \pmod{2q+2}$

$= 2\,[1+2+\ldots.+q] \pmod{2q+2}$

$= 2\,\dfrac{q(q+1)}{2} \pmod{2q+2}$

$=(q^2+q) \pmod{2q+2}$ …… (2)

From (1) and (2), we get

$4qk-2q+2dq^2 = (q^2+q) \pmod{2q+2}$

(i.e.) $4qk-2q+2dq^2-q^2-q \equiv 0 \pmod{2q+2}$

(i.e.) $4qk-3q+q^2(2d-1) \equiv 0 \pmod{2q+2}$

Hence the lemma.

Theorem 2.14

If $p = 2k-1$ and p is odd then there is no (k, d)-*EEGL* for any tree T.

Proof

Given $p = 2k-1$

(i.e.) $q+1 = 2k-1$

$2k = q+2$

$4k = 2q+4$

From lemma 2.13,

$4qk-3q+q^2(2d-1) \equiv 0 \pmod{2q+2}$

$q(2q+4)-3q+q^2(2d-1) \equiv 0 \pmod{2q+2}$

$q(2q+2)+2q-3q+q^2(2d-1) \equiv 0 \pmod{2q+2}$

$2q^2d-q-q^2 \equiv 0 \pmod{2q+2}$

$2q^2d+2qd-2qd-q-q^2 \equiv 0 \pmod{2q+2}$

$(2q+2)\,qd-2qd-q-q^2 \equiv 0 \pmod{2q+2}$

$-2qd-q-q^2 \equiv 0 \pmod{2q+2}$

$(-2q-2)\,d+2d-q-q^2 \equiv 0 \pmod{2q+2}$

$2d-q(q+1) \equiv 0 \pmod{2q+2}$

$d-\dfrac{q(q+1)}{2} \equiv 0 \pmod{q+1}$

$d \equiv 0 \pmod{p}$

$d \equiv 0 \pmod{p}$

$\Rightarrow$ p / d

$\Rightarrow$ $d = p, 3p, 5p, \ldots$ (since p is odd.)

Thus by observation 2.12 (1),

$f(E) = \{2p, 4p, 6p, \ldots\}$ and hence for any vertex v, $f^{+}(v) = 0$.

Therefore, if $p = 2k-1$ then there is no *(k, d) - EEGL* for any tree T.

Corollary 2.15

If $p = 2k-1$ and p is odd then there is no $(lp+k, d)$-*EEGL* for any tree T.

Proof

Let $p = 2k-1$. Then by Theorem 2.14 there is no (k, d)-*EEGL* for any tree T.

By Corollary 2.6 there is no $(lp+k, d)$-*EEGL* for any tree T.

Illustration 2.16

$B_{2,3}$ has no $(4, d)$ - *EEGL*.

For, if possible let f be a (k, d) - *EEGL* for $B_{2,3}$

Here $k = 4$; $p = 7 = 2k-1$ and $q = 6$.

By Lemma 2.13,

$4qk-3q+q^2(2d-1) \equiv 0 \pmod{2q+2}$

$4 \times 6 \times 4 - 3 \times 6 + 6^2 (2d - 1)$	$\equiv 0$	(mod 14)
$64 - 18 + 36 (2d - 1)$	$\equiv 0$	(mod 14)
$72d$	$\equiv 0$	(mod 14)
$36d$	$\equiv 0$	(mod 7)
d	$\equiv 0$	(mod 7) but d is odd.

Therefore $d = 7, 21, \ldots$

By observation 2.12 (1),

$f(E) = \{14, 28, 42, \ldots\}$ and hence for any vertex v, $f^+(v) = 0$ (mod 14).

This is not possible.

Therefore there is no $(4, d)$ - *EEGL* for $B_{2,3}$.

Note 2.17

The above illustration means that there is no $(7l + 4, d)$ - *EEGL* for $B_{2,3}$ where $l \geq 0$, by Corollary 2.15.

Corollary 2.18

$B_{n,m}$ has no (k, d) - *EEGL* for $k = \dfrac{n+m+3}{2}$ for any d.

Proof

For, $2k - 1 = n + m + 2 = p$.

Then by Theorem 2.14 there is no (k, d) - *EEGL* for any tree T.

Therefore $B_{n,m}$ has no (k, d) - *EEGL* for $k = \dfrac{n+m+3}{2}$ for any d.

Observation 2.19

There is no (k,p) - *EEGL* for any tree T of odd order p where $k \geq 1$.

Proof

Suppose there is a (k,p) - *EEGL* f for T

Then $f(E) = \{2k - 1 + p, 2k - 1 + 3p, 2k - 1 + 5p, \ldots, 2k - 1 + (2q - 1) p\}$.

But all these values are congruent to $2k - 1 + p$ (mod $2p$). But there are atleast two pendant vertices u and v in T

Therefore $f^+(u) = f^+(v) = 2k - 1 + p$ (mod $2p$)

This is not possible. Therefore is there is no (k,p) - *EEGL* for any tree T of odd order p where $k \geq 1$.

Observation 2.20

There is no $(k, (2l+1)\, p)$ - *EEGL* for any tree T of odd order p where l *and* $k \geq 1$.

Proof

By Corollary 2.10, if there is no (k, d) - *EEGL* for any tree T of odd order p, then there is no $(k, 2lp + d)$ - *EEGL* for T where l, k and $d \geq 1$. Therefore from Observation 2.19, there is no $(k, (2l+1)\, p)$ - *EEGL* for any tree T of odd order p where l *and* $k \geq 1$.

Illustration 2.21

$T(11,10)$is a tree with $p = 11$ and $q = 10$.

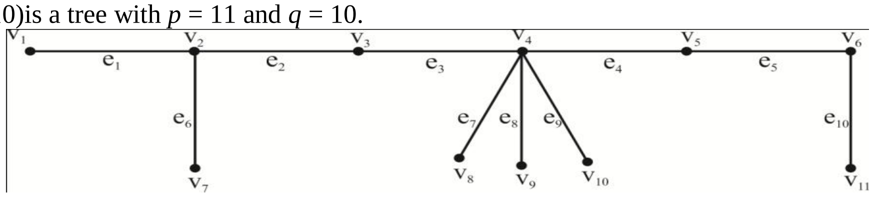


Figure 6 : T has no $(k, (2l+1)\, 7)$ -*EEGL* where $l \geq 0$ and $k \geq 1$.

Illustration 2.22

T (17, 16) is a tree with p =17 and q =16.

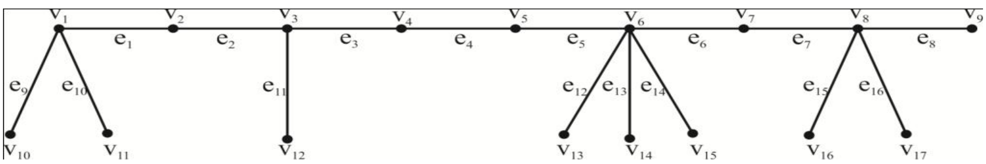


Figure 7 : *T* has no $(k, (2l+1)\, 13)$ - *EEGL* where $l \geq 0$ and $k \geq 1$.

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