International Journal of Engineering Science Invention
ISSN (Online): 2319 – 6734, ISSN (Print): 2319 – 6726
www.ijesi.org Volume 2 Issue 1 ‖ January. 2013 ‖ PP.21-26

L-fuzzy sub ℓ-group and its level sub ℓ-groups

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ABSTRACT: *In this paper, we discussed some properties of L-fuzzy sub ℓ-group of a lattice ordered group and define a new algebraic structure of an anti L-fuzzy sub ℓ-group and some related properties are investigated. We establish the relation between L-fuzzy sub ℓ-group and anti L-fuzzy sub ℓ-group. The purpose of this study is to implement the fuzzy set theory and group theory in L-fuzzy sub ℓ-group and anti L-fuzzy sub ℓ-groups. Characterizations of level subsets of a L-fuzzy sub ℓ-group are given. We also discussed the relation between a given a L-fuzzy sub ℓ-group and its level sub ℓ-groups and investigate the conditions under which a given sub ℓ-group has a properly inclusive chain of sub ℓ-groups. In particular, we formulate how to structure an L-fuzzy sub ℓ-group by a given chain of sub ℓ-groups.*

Keywords— *Fuzzy set, Fuzzy sub group, L-fuzzy sub l-group of a group, Anti L-Fuzzy Sub l-Group of a group. AMS Subject Classification (2000): 20N25, 03E72, 03F055, 06F35, 03G25.*

I. INTRODUCTION

L. A. Zadeh [14] introduced the notion of a fuzzy subset A of a set X as a function from X into [0, 1]. Rosenfeld [8] applied this concept in group theory and semi group theory, and developed the theory of fuzzy subgroups and fuzzy sub semi groupoids respectively. J.A. Goguen [2] replaced the valuations set [0, 1], by means of a complete lattice in an attempt to make a generalized study of fuzzy set theory by studying L-fuzzy sets. In fact it seems in order to obtain a complete analogy of crisp mathematics in terms of fuzzy mathematics, it is necessary to replace the valuation set by a system having more rich algebraic structure. These concepts ℓ-groups play a major role in mathematics and fuzzy mathematics. Satya Saibaba[13] introduced the concept of L- fuzzy ℓ-group and L-fuzzy ℓ-ideal of ℓ-group.

We wish to define a new algebraic structure of L-fuzzy sub ℓ-group and establishes the relation with L-fuzzy sub ℓ- group and discussed some of its properties.

II. PRELIMINARIES

In this section we site the fundamental definitions that we will be used in the sequel. Throughout this paper $G = (G,*)$ is a group, e is the identity element of G and xy we mean x*y.

2.1 Definition [13]

A lattice ordered group or a ℓ-group is a system G = (G, ∗ , ≤), where

i. (G, ∗) is a group,
ii. (G, ≤) is a lattice ,
iii. The inclusion is invariant under all translations $x \rightarrow a+x+b$ i.e. $x \leq y \Rightarrow a+x+b \leq a+y+b$

Remark

Throughout this paper G = (G, ∗ , ≤) is a lattice ordered group or a ℓ-group, e is the identity element of G and xy we mean x∗y.

2.2 Definition [8]

Let S be any non-empty set. A fuzzy subset μ of S is a function μ: S → [0 ,1].

2.3 Definition [8]

Let G be a group. A fuzzy subset μ of G is called a fuzzy subgroup if for any x , y ∈ G,

i. $\mu(xy) \geq \mu(x) \wedge \mu(y)$,
ii. $\mu(x^{-1}) = \mu(x)$.

2.4 Definition [1]

Let G be a group. A fuzzy subset μ of G is called an anti fuzzy subgroup if for any x , y $\in$ G,

i. $\mu(xy) \leq \mu(x) \vee \mu(y)$,

ii. $\mu(x^{-1}) = \mu(x)$.

2.5 Definition [13]

An L the infinite meets distributive law. If L is the unit interval [0, 1] of real numbers, there are the usual fuzzy subsets of G. A L fuzzy subset μ: G $\rightarrow$ Lis said to be non-empty, if it is not the constant map which assumes the values 0 of L.

2.6 Definition [11]

A L-fuzzy subset μ of G is said to be a L-fuzzy subgroup of G if for any x , y $\in$ G,

i. $\mu(xy) \geq \mu(x) \wedge \mu(y)$,

ii. $\mu(x^{-1}) = \mu(x)$.

2.7 Definition [11]

A L-fuzzy subset μ of G is said to be an anti L-fuzzy subgroup of G if for any x , y $\in$ G,

i. $\mu(xy) \leq \mu(x) \vee \mu(y)$,

ii. $\mu(x^{-1}) = \mu(x)$.

2.8 Definition[13]

A L-fuzzy subset μ of G is said to be a L-fuzzy sub l group of G if for any x , y $\in$ G,

i. $\mu(xy) \geq \mu(x) \wedge \mu(y)$,

ii. $\mu(x^{-1}) = \mu(x)$,

iii. $\mu(x \vee y) \geq \mu(x) \wedge \mu(y)$,

iv. $\mu(x \wedge y) \geq \mu(x) \wedge \mu(y)$.

2.9 Definition [13]

A L-fuzzy subset μ of G is said to be an anti L-fuzzy sub l group of G if for any x , y $\in$ G,

i. $\mu(xy) \leq \mu(x) \vee \mu(y)$,

ii. $\mu(x^{-1}) = \mu(x)$,

iii. $\mu(x \vee y) \leq \mu(x) \vee \mu(y)$,

iv. $\mu(x \wedge y) \leq \mu(x) \vee \mu(y)$.

III. PROPERTIES OF L-FUZZY SUB l -GROUP OF G

In this section we discuss some of the properties of L-fuzzy sub l -group of G.

3.1 Theorem

Let μ be a L-fuzzy sub ℓ -group of G then ,

i. $\mu(x) \leq \mu(e)$ for x $\in$ G, where e is the identity element of the G.

ii. The Subset H = { x $\in$ G / $\mu(x) = \mu(e)$ } is a sub ℓ -group of G.

Proof

i. Let x $\in$ G, then

$\mu(x) = \mu(x) \wedge \mu(x)$,
$\mu(x) = \mu(x) \wedge \mu(x^{-1})$,
$\leq \mu(xx^{-1})$,
$= \mu(e)$.

That is, $\mu(x) \leq \mu(e)$.

ii. Let H = { x $\in$ G / $\mu(x) = \mu(e)$ }.

Clearly H is non-empty as e $\in$ H and for any x , y $\in$ G, we have,
$\mu(x) = \mu(y) = \mu(e)$.

Now, $\mu(xy^{-1}) \geq \mu(x) \wedge \mu(y^{-1})$,
$= \mu(x) \wedge \mu(y)$,
$= \mu(e) \wedge \mu(e)$,

That is , $\mu(xy^{-1}) \geq \mu(e)$ and obviously $\mu(xy^{-1}) \leq \mu(e)$, by (i).

Hence, $\mu(xy^{-1}) = \mu(e)$ and $xy^{-1} \in$ H.

Hence , H is a sub ℓ -group of a group G.

3.2 Theorem

Let μ be a L-fuzzy sub ℓ -group of G with identity e , then, for any x , y $\in$ G, $\mu(xy^{-1}) = \mu(e) \Rightarrow \mu(x) = \mu(y)$.

Proof

Let μ be a L-fuzzy sub ℓ -group of G with identity e and $\mu(xy^{-1}) = \mu(e)$, then for for any x , y $\in$ G, we have,

$$\begin{aligned} \mu(x) &= \mu(x(y^{-1}y)) \\ &= \mu((xy^{-1})\, y) \\ &\geq \mu(xy^{-1}) \wedge \mu(y) \\ &= \mu(e) \wedge \mu(y) \\ &= \mu(y). \end{aligned}$$

That is, $\mu(x) \geq \mu(y)$.

Now, $\mu(y) = \mu(y^{-1})$, then

$$\begin{aligned} &= \mu(ey^{-1}) \\ &= \mu((x^{-1}x)y^{-1}) \\ &= \mu(x^{-1}(xy^{-1})) \\ &\geq \mu(x^{-1}) \wedge \mu(xy^{-1}) \\ &\geq \mu(x) \wedge \mu(e) \end{aligned}$$

$\mu(y) \geq \mu(x)$.

Hence, $\mu(x) = \mu(y)$.

3.3 Theorem

Let μ be a L-fuzzy sub ℓ -group of G iff $\mu(xy^{-1}) \geq \mu(x) \wedge \mu(y)$ for any x , y $\in$ G.

Proof

Let μ be a L-fuzzy sub ℓ -group of G, then for any x , y $\in$ G , we have

$$\mu(xy) \geq \mu(x) \wedge \mu(y),$$

Now, $\mu(xy^{-1}) \geq \mu(x) \wedge \mu(y^{-1})$,

$$= \mu(x) \wedge \mu(y),$$

$\Leftrightarrow \quad \mu(xy^{-1}) \geq \mu(x) \wedge \mu(y)$.

3.4 Theorem

Let μ and λ be any two L-fuzzy sub ℓ -group of G, then $\mu\cap\lambda$ is a L-fuzzy sub ℓ -group of G.

Proof

Let μ and λ be an L-fuzzy sub l -group of G.

i. $$\begin{aligned} (\mu\cap\lambda)(xy) &= \mu(xy) \wedge \lambda(xy) \\ &\geq (\mu(x) \wedge \mu(y)) \wedge (\lambda(x) \wedge \lambda(y)) \\ &= ((\mu(x) \wedge \lambda(x)) \wedge (\mu(y) \wedge \lambda(y)) \\ &= (\mu\cap\lambda)(x) \wedge (\mu\cap\lambda)(y). \end{aligned}$$

$(\mu\cap\lambda)(xy) \geq (\mu\cap\lambda)(x) \wedge (\mu\cap\lambda)(y)$.

ii. $(\mu\cap\lambda)(x^{-1}) = \mu(x^{-1}) \wedge \lambda(x^{-1})$

$= \mu(x) \wedge \lambda(x)$

$= (\mu\cap\lambda)(x)$.

$(\mu\cap\lambda)(x^{-1}) = (\mu\cap\lambda)(x)$

iii. $$\begin{aligned} (\mu\cap\lambda)(x\vee y) &= \mu(x\vee y) \wedge \lambda(x\vee y) \\ &\geq (\mu(x) \wedge \mu(y)) \wedge (\lambda(x) \wedge \lambda(y)) \\ &= ((\mu(x) \wedge \lambda(x)) \wedge (\mu(y) \wedge \lambda(y)) \\ &= (\mu\cap\lambda)(x) \wedge (\mu\cap\lambda)(y). \end{aligned}$$

$(\mu\cap\lambda)(x\vee y) \geq (\mu\cap\lambda)(x) \wedge (\mu\cap\lambda)(y)$.

iv. $$\begin{aligned} (\mu\cap\lambda)(x\wedge y) &= \mu(x\wedge y) \wedge \lambda(x\wedge y) \\ &\geq (\mu(x) \wedge \mu(y)) \wedge (\lambda(x) \wedge \lambda(y)) \\ &= ((\mu(x) \wedge \lambda(x)) \wedge (\mu(y) \wedge \lambda(y)) \\ &= (\mu\cap\lambda)(x) \wedge (\mu\cap\lambda)(y). \end{aligned}$$

$(\mu \cap \lambda)(x \wedge y) \geq (\mu \cap \lambda)(x) \wedge (\mu \cap \lambda)(y)$.

Hence, $\mu \cap \lambda$ is a L-fuzzy sub ℓ -group of G.

Remark

Arbitrary intersection of a L-fuzzy sub ℓ -group of G is a L- fuzzy sub ℓ -group of G.

3.5 Theorem

μ is a L-fuzzy sub ℓ -group of G iff μ^c is an anti L-fuzzy sub ℓ -group of G.

Proof

Let μ be a L-fuzzy sub ℓ -group of G and for x , y $\in$ G, we have

i. $\mu(xy) \geq \mu(x) \wedge \mu(y)$

$\Leftrightarrow 1 - \mu^c(xy) \geq (1-\mu^c(x)) \wedge (1-\mu^c(y))$

$\Leftrightarrow \mu^c(xy) \leq 1- (1-\mu^c(x)) \wedge (1-\mu^c(y))$

$\Leftrightarrow \mu^c(xy) \leq \mu^c(x) \vee \mu^c(y)$.

ii. $\mu(x^{-1}) = \mu(x)$ for any x , y $\in$ G.

$\Leftrightarrow 1 - \mu^c(x^{-1}) = 1- \mu^c(x)$

$\Leftrightarrow \mu^c(x^{-1}) = \mu^c(x)$.

iii. $\mu(x \vee y) \geq \mu(x) \wedge \mu(y)$

$\Leftrightarrow 1-\mu^c(x \vee y) \geq (1-\mu^c(x)) \wedge (1-\mu^c(y))$

$\Leftrightarrow \mu^c(x \vee y) \leq 1- (1-\mu^c(x)) \wedge (1-\mu^c(y))$

$\Leftrightarrow \mu^c(x \vee y) \leq \mu^c(x) \vee \mu^c(y)$.

iv. $\mu(x \wedge y) \geq \mu(x) \wedge \mu(y)$

$\Leftrightarrow 1-\mu^c(x \wedge y) \geq (1-\mu^c(x)) \wedge (1-\mu^c(y))$

$\Leftrightarrow \mu^c(x \wedge y) \leq 1- (1-\mu^c(x)) \wedge (1-\mu^c(y))$

$\Leftrightarrow \mu^c(x \wedge y) \leq \mu^c(x) \vee \mu^c(y)$.

Hence, μ^c is an anti L-fuzzy sub ℓ -group of G.

IV. PROPERTIES OF LEVEL SUBSETS OF A L-FUZZY SUB ℓ -GROUP OF G

In this section, we introduce the concept of level subsets of a L-fuzzy sub ℓ -group of G and discussed some of its properties.

4.1 Definition

Let μ be a L-fuzzy sub ℓ -group of G. For any t $\in$[0,1], we define the set
U (μ ; t) = { x$\in$G / $\mu(x) \geq t$ } is called a upper level subset or a level subset of μ.

4.1 Theorem

Let μ be a L-fuzzy sub ℓ -group of G. Then for t $\in$ [0, 1] such that $t \leq \mu(e)$, U (μ ; t) is a sub ℓ -group of G.

Proof

For any x, y $\in$ U (μ ; t) , we have,

$\mu(x) \geq t$; $\mu(y) \geq t$.

Now, $\mu(xy^{-1}) \geq \mu(A) \wedge \mu(B)$

$\mu(xy^{-1}) \geq t \wedge t$

$\mu(xy^{-1}) \geq t$.

$xy^{-1} \in$ U (μ ; t).

Hence, U (μ ; t) is a sub ℓ -group of ϑ.

4.2 Theorem

Let μ be a L-fuzzy subset of G such that U(μ ; t) is a sub ℓ -group of G. For t $\in$ [0,1] $t \leq \mu(e)$, μ is a L-fuzzy sub ℓ -group of G.

Proof

Let x, y $\in$ G and $\mu(x) = t_1$ and $\mu(y) = t_2$.

Suppose $t_1 < t_2$, then x , y $\in$ U(μ ; t_1).

As U (μ ; t_1) is a sub ℓ -group of G, $xy^{-1} \in$ U(μ ; t_1).

Hence, $\mu(xy^{-1}) \geq t_1 = t_1 \wedge t_2$

$\geq \mu(x) \wedge \mu(y)$

That is, $\mu(xy^{-1}) \geq \mu(x) \wedge \mu(y)$.

By Theorem 3.2, μ is a L-fuzzy sub ℓ-group of G.

4.2 Definition

Let μ be a L-fuzzy sub ℓ-group of G. The sub ℓ-groups U (μ ; t) for $t \in [0,1]$ and $t \leq \mu(e)$, are called level sub ℓ-groups of μ.

4.3 Theorem

Let μ be a L-fuzzy sub ℓ-group of G. If two level sub ℓ-groups $U(\mu ; t_1)$, $U(\mu ; t_2)$, for, $t_1, t_2 \in [0,1]$ and $t_1, t_2 \leq \mu(e)$ with $t_1 < t_2$ of μ are equal then there is no $x \in G$ such that $t_1 \leq \mu(x) < t_2$.

Proof

Let $U(\mu ; t_1) = U(\mu; t_2)$.

Suppose there exists $x \in G$ such that $t_1 \leq \mu(x) < t_2$, then $U(\mu ; t_2) \subseteq U(\mu ; t_1)$.

Then $x \in U(\mu ; t_1)$, but $x \notin U(\mu ; t_2)$, which contradicts the assumption that, $U(\mu ; t_1) = U(\mu ; t_2)$. Hence there is no $x \in G$ such that $t_1 \leq \mu(x) < t_2$.

Conversely, suppose that there is no $x \in G$ such that $t_1 \leq \mu(x) < t_2$,

Then, by definition, $U(\mu ; t_2) \subseteq U(\mu ; t_1)$.

Let $x \in U(\mu ; t_1)$ and there is no $x \in G$ such that $t_1 \leq \mu(x) < t_2$.

Hence, $x \in U(\mu ; t_2)$ and $U(\mu ; t_1) \subseteq U(\mu ; t_2)$.

Hence, $U(\mu ; t_1) = U(\mu ; t_2)$.

4.4 Theorem

A L-fuzzy subset μ of G is a L-fuzzy sub ℓ-group of G if and only if the level subsets U(μ ; t), $t \in$ Image μ, are sub ℓ-groups of ϑ.

Proof It is clear.

4.5 Theorem

Any sub ℓ-group H of G can be realized as a level sub ℓ-group of some L-fuzzy sub ℓ-group of G.

Proof

Let μ be a L-fuzzy subset and $x \in G$.

Define,

$$\mu(x) = \begin{cases} t & \text{if } x \in H, \text{ where } t \in (0,1]. \\ 0 & \text{if } x \notin H, \end{cases}$$

We shall prove that μ is a L-fuzzy sub ℓ-group of G.

Let $x, y \in G$.

i. Suppose $x, y \in H$, then $xy \in H$, $xy^{-1} \in H$, $x \vee y \in H$ and $x \wedge y \in H$.
$\mu(x) = t$, $\mu(y) = t$, $\mu(xy^{-1}) = t$, $\mu(x \vee y) = t$ and $\mu(x \wedge y) = t$.
Hence $\mu(xy^{-1}) \geq \mu(x) \wedge \mu(y)$
$\mu(x \vee y) \geq \mu(x) \wedge \mu(y)$,
$\mu(x \wedge y) \geq \mu(x) \wedge \mu(y)$.

ii. Suppose $x \in H$ and $y \notin H$, then $xy \notin H$ and $xy^{-1} \notin H$.
$\mu(x) = t$, $\mu(y) = 0$ and $\mu(xy^{-1}) = 0$.
Hence $\mu(xy^{-1}) \geq \mu(x) \wedge \mu(y)$.

iii. Suppose $x, y \notin H$, then $xy^{-1} \in H$ or $xy^{-1} \notin H$.
$\mu(x) = 0$, $\mu(y) = 0$ and $\mu(xy^{-1}) = t$ or 0.
Hence $\mu(xy^{-1}) \geq \mu(x) \wedge \mu(y)$.

Thus in all cases, μ is a L-fuzzy sub ℓ-group of G.

For this L-fuzzy sub ℓ-group, $U(\mu ; t) = H$.

Remark

As a consequence of the **Theorem 4.3 and Theorem 4.4**, the level sub ℓ-groups of a L-fuzzy sub ℓ-group μ of G form a chain. Since $\mu(e) \geq \mu(x)$ for any $x \in G$ and therefore, $U(\mu ; t_0)$, where $\mu(e) = t_0$ is the smallest and we have the chain :

$\{e\}= U(\mu ; t_0) \subset U(\mu ; t_1) \subset U(\mu ; t_2) \subset \ldots \subset U(\mu ; t_n) = \vartheta$, where $t_0 > t_1 > t_2 > \ldots\ldots > t_n$.

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