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Analysis of compartment model for phosphorus in lake water and the sediment: Approach to fuzzy differential equation

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In this paper, a phosphorus in lake water and sediment model has been considered in fuzzy environment. The proposed model contains two compartments such as compartment 1($c_1(t)$)and compartment 2 ($c_2(t)$) and the initial values of the model are assumed by fuzzy number (Triangular fuzzy number). We have used Hukuhara differentiable technique to convert fuzzy phosphorus in lake water model into a system of differential equations. The feasible equilibrium points, the stability conditions have been sightseen in this paper. All fuzzy solutions of the are strong or weak fuzzy solutions have been verified. In order to validate the outcomes of the model, numerical simulations are carried out using the MATLAB program.

***Keywords**: Phosphorus model in lake water, Triangular fuzzy number, Fuzzy differential equation (FDE), Local stability, Strong or weak fuzzy solutions.*

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I. Introduction:

Recently CSTR have been utilized more widely to deal with various type of problems. In biological and environmental engineering, compartmental models are commonly used to explore a wide range of problems. Let us examine the phenomenon of phosphorus release in lake water utilizing a compartmental model. Eutrophication results from the influx of nitrogen- and phosphorus-containing compounds into water bodies as a consequence of human activities. This process leads to excessive growth of Phytoplankton, Periphyton, along with macrophytes, causing issues in lake water, including diminished dissolved oxygen, unpleasant odors, discoloration, and adverse effects on aquatic life. Suppose the lake is exposed to sewerage that contains phosphates and part of the phosphate content precipitates and deposits with the sediments. In stratified conditions, less oxygen penetrates to the bottom water, resulting in mostly anaerobic conditions in the sediment, where phosphorus is concurrently exposed into the lake water ('sediment feedback'). Some researchers dedicated their attention to modeling the discharge of phosphate in the lake water. Chapra and Canale (1991) [1] developed a two-compartment model to describe the phosphate content in the sediment layer at the bottom of the lake.

Many problems develop as a result of the phosphorus emission into lake water. In recent decades, researchers have focused on developing compartmental models to represent lake water issues caused by phosphorus discharge [2-8]. Gentleman 2002 [9] describes marine plankton models. However, no action has been made so far to mathematically explain the phosphorus disposal model into lake water.

Uncertainty is a key consideration when studying phosphorus release in lake sediments. Uncertainty enters the biological and environmental systems as a result of a lack of data, insufficient data supply, technology faults, and environmental fluctuations. To deal with this form of chaos in the environment, researchers are now working on developing a mathematical model describing phosphorus free in lake sediments using the FDEs technique to obtain a realistic scenario.

Many disciplines have investigated fuzzy differential equations, including engineering, biology, physics, etc [10-19]. In [20], the optimal use of renewable resources by populations with uncertain characteristics is detailed. A crisp and fuzzy environment describes the dynamic behavior of the malaria model [21]. In an uncertain environment, the dynamic behavior of a prey-predator model with an MSY policy is detailed in relation to various harvesting strategies [22]. An essential function of fuzzy set theory is to simulate many kinds of real-world scenarios. Fuzzy sets and fuzzy numbers are big topics right now. The original authors of fuzzy set theory are Zadeh [23] and Dubois and Parade [24].

Intuitionistic fuzzy set theory (IFST) was developed as an extension of fuzzy set theory (FST) [25]. It was initially proposed by Atanassov [26,27] as an advancement of the FST originally proposed by Zadeh [23] and later expanded by Dubois and Prade [24]. The IFST approach is particularly suitable for modeling uncertainty and hesitation. While FST assumed the degree of membership, IFST further incorporates uncertainty by accounting for both membership and non-membership degrees [28].

IFST(intuitionistic fuzzy set theory) is one of the most applicable of last few years. IFST is uses in different branch of science mainly used in robotics, transmission energy, and audiovidual system etc. Atanassov[25-27] first utilize the concept IFST. Mohamed Melliani [29] introduced the concept of metric space in IFS. Later, Mohamed Melliani [30] proved the uniqueness and exitence theorem for the solution of an intuitionistic fuzzy differential equation with a nonlocal condition. Many researchers have devoted considerable effort to the numerical study of intuitionistic fuzzy differential equations [31–34]. Also, fuzzy diff. equation plays an important role of characterization of different biological models and the characteristics equation. Some attractive methodology and theories based on fuzzy valued derivative include some interesting part of FDE[35-41] have been published on the purpose of characterization theorem to solution of FDE. we can enhance our model to utilize neutrosophic fuzzy set theory[NFST], as its one of the generalized form of FST[42-45].

We have made an attempt to provide fascinating contributions through the use of novel clues and skills, which could produce a new and improved edition. In this paper, we improve the functional notions and methodologies as

(i) The phosphorus model system is discussed using fuzzy set.
(ii) The model's initial state is typically represented by triangular fuzzy set.
(iii) This paper discusses the process of converting a fuzzy of α-cut.
(iv) This paper examines how to analyze the stability of a fuzzy phosphorus model.
(v) Proposed model is the demonstrated using the fuzzy differential equation (NFDE). Different sorts of generalized fuzzy derivatives helped to establish FDE.

In this paper has the following structure: The first portion includes a basic introduction. Section 2 contains useful prerequisite topics. Section 3 contains the mathematical model formulation for phosphorus compartment. Section 4 discusses the phosphorus model in a fuzzy environment. Section 5 now includes numerical verification of the theoretical concepts covered. Finally, section 6 addresses the work's conclusion.

II. Preliminary concept:

Definition 2.1 Fuzzy set [46]:
Let Y_{fuzz} is a set(non-empty). An FS $\tilde{T}_{fuzz}$ on Y_{fuzz} can be described as $\tilde{T}_{fuzz}$= {$\bar{y}_{\varsigma}$,M($\bar{y}_{\varsigma}$,):$\bar{y}_{\varsigma}$,∈ Y_{fuzz} } . whereas M($\bar{y}_{\varsigma}$): Y_{fuzz} to [0,1] represent the degree of membership of the fuzzy set $\tilde{T}_{fuzz}$.

Definition 2.2 α-cut of a fuzzy number[FN] [46]:
α -cut of FN $\widetilde{M_{fuzz}}$ can be described in crisp set as follows-
M_{α} ={$\bar{y}_{\varsigma}|u_{\tilde{M}}(\bar{y}_{\varsigma})$ $greater\ than\ or\ equalto\ \alpha$} where $\alpha \in [0,1]$. Where $M_{\alpha}$is a closed bounded non-void set in Y_{fuzz}, $M_{LT}(\alpha)$, $M_{RT}(\alpha)$ two intervals as upper bounds and lower bounds interval respectively and it can described by, M_{α}=[$M_{LT}(\alpha)$, $M_{RT}(\alpha)$].
The α -cut of triangular fuzzy number $\widetilde{M_{\alpha}}$=$(a_{\sigma1}, a_{\sigma2}, a_{\sigma3})$ can be exposed as =[$M_{\varsigma}^{LT}(\alpha)$, $M_{\varsigma}^{RT}(\alpha)$] where, $M_{\varsigma}^{LT}(\alpha) = a_{\sigma1} + \alpha(a_{\sigma2} - a_{\sigma1})$, $M_{\varsigma}^{RT}(\alpha) = a_{\sigma3} - \alpha(a_{\sigma3} - a_{\sigma2})$ ∀ $\alpha \in [0,1]$.

Definition 2.3 Triangular fuzzy number [46]:
A triangular fuzzy number (TFN) described by $\widetilde{M_\alpha} = (a_{\sigma1}, a_{\sigma2}, a_{\sigma3})$ and the function of membership determine as-

$$\mu_{\widetilde{M}}(y_\varsigma) = \begin{cases} \frac{1}{a_{\sigma2} - a_{\sigma1}}(y_\varsigma - a_{\sigma1}), a_{\sigma1} \le y_\varsigma < a_{\sigma2} \\ 1, \quad y_\varsigma = a_{\sigma2} \\ \frac{1}{a_{\sigma3} - a_{\sigma2}}(a_{\sigma3} - y_\varsigma), a_{\sigma2} < y_\varsigma \le a_3 \\ 0, otherwise \end{cases}$$

Definition 2.4 Generalised Hukuhara fuzzy derivative [46]:
The gHD-(generalised H-derivative)of fuzzy valued function $\Theta_f: (m, n)$ ***to R*** at t_0 is expressed by

$$\Theta'_f(t_0) = \lim_{\varepsilon \to 0} \frac{\Theta_f(t_0 + \varepsilon) - \Theta_f(t_0)}{\varepsilon}$$

If $\Theta'_f(t_0) \in \mathcal{R}$ fulfil the condition (2.1), then we remarks that $\Theta'_f(t_0)$ is generalised H-differentiable at t_0.
And, we state that $\Theta_f(t_0)$ is (i)-gHD at t_0 if $[\Theta'_f(t_0)]_\alpha = [\Theta'_{Lf}(t_0,\alpha), \Theta'_{Rf}(t_0,\alpha)]$ also, $\Theta_f(t_0)$ is (ii)-gHD at t_0 if $[\Theta'_f(t_0)]_\alpha = [\Theta'_{Rf}(t_0,\alpha), \Theta'_{Lf}(t_0,\alpha)]$
Definition 2.5 Strong and weak solution of FDE [46]: Assume the FDE(fuzzy- differential- equation) $\frac{d\widetilde{y_\xi}(t)}{dt} = g_1\left(\tilde{u}, \widetilde{y_\xi}(t)\right)$ with $\widetilde{y_\xi}(t_0) = \tilde{y}_0$, $\tilde{u}$ and (or) $\widetilde{y_\xi}(t)$ are the fuzzy numbers.
Let, $\widetilde{y_\xi}(t)$ is the solution of FDE and the α −cut be $\widetilde{y_\xi}(t,\alpha) = \left(y_{\xi1}(t,\alpha), y_{\xi2}(t,\alpha)\right)$. $\tilde{y}(t,\alpha)$ is described as strong solution if $y_{\xi1}(t,\alpha) \le y_{\xi2}(t,\alpha)\ \forall\ \alpha \in [0,1]$. Otherwise $\widetilde{y_\xi}(t,\alpha)$ is a solution(weak). We can transform the solution of weak to strong as:

$$\widetilde{y_\xi}(t,\alpha) = \left[min\left(y_{\xi1}(t,\alpha), y_{\xi2}(t,\alpha)\right), max\left(y_{\xi1}(t,\alpha), y_{\xi2}(t,\alpha)\right)\right]$$

III. Model formulation:

Phosphorus loads occur at a rate of $W_{\tilde{\eta}}$ kg/year through the influent water. Some phosphorus is lost in the flush water, and the lake is flushed at a rate of $Q_{\tilde{\eta}}$ (cm^3/year). A setting velocity (in meters per year) and an area (in square meters) of sediment are used to measure the transport of phosphorus compartment 1 to the surface sediment. Phosphorus is recycled from surface sediment into water at a rate of one million cubic meters per year, with no change to the area of mass transfer, denoted as $A_{\tilde{\eta}}\ m^2$. Phosphorus burial takes place in deep sediments with a burial mass transfer coefficient K_b.

Two compartment phosphorus model [47] can be written as,

$$v_{\eta_1}\frac{dc_1(t)}{dt} = W_{\tilde{\eta}} - Q_{\tilde{\eta}}c_1(t) - V_sAc_1(t) + K_sA_{\tilde{\eta}}c_2(t)$$
$$v_{\eta_2}\frac{dc_2(t)}{dt} = V_sA_{\tilde{\eta}}c_1(t) - K_sA_{\tilde{\eta}}c_2(t) - K_bA_{\tilde{\eta}}c_2(t) \qquad (1)$$

With the initial condition, $c_1(t_0) = c_{01}, c_2(t_0) = c_{02}$

An amount v_1 of water that has a phosphorus level of c_1 is present in Compartment 1. The second compartment is made up of the "surface sediment." It has a "lumped" phosphorus concentration (c_2) and a volume (v_2).
The above system (1) can be rewritten as follows:

$$\frac{dc_1(t)}{dt} = \frac{W_{\tilde{\eta}}}{v_{\eta_1}} - \frac{Q_{\tilde{\eta}}c_1(t)}{v_{\eta_1}} - \frac{V_s A_{\tilde{\eta}}c_1(t)}{v_{\eta_1}}c_1(t) + \frac{K_sA_{\tilde{\eta}}c_2(t)}{v_{\eta_1}}$$
$$\frac{dc_2(t)}{dt} = \frac{V_sA_{\tilde{\eta}}c_1(t)}{v_{\eta_2}} - \frac{K_sA_{\tilde{\eta}}c_2(t)}{v_{\eta_2}} - \frac{K_bA_{\tilde{\eta}}c_2(t)}{v_{\eta_2}} \qquad (2)$$

with the initial condition, $c_1(t_0) = c_{01}, c_2(t_0) = c_{02}$
The above crisp model converted into fuzzy environment given as follows:

$$\frac{d\widetilde{c_1}(t)}{dt} = \frac{W_{\tilde{\eta}}}{v_{\eta_1}} - \frac{Q_{\tilde{\eta}}\widetilde{c_1}(t)}{v_{\eta_1}} - \frac{V_s A_{\tilde{\eta}}\widetilde{c_1}(t)}{v_{\eta_1}} + \frac{K_sA_{\tilde{\eta}}\widetilde{c_2}(t)}{v_{\eta_1}}$$
$$\frac{d\widetilde{c_2}(t)}{dt} = \frac{V_sA_{\tilde{\eta}}\widetilde{c_1}(t)}{v_{\eta_2}} - \frac{K_sA_{\tilde{\eta}}\widetilde{c_2}(t)}{v_{\eta_2}} - \frac{K_bA_{\tilde{\eta}}\widetilde{c_2}(t)}{v_{\eta_2}} \qquad (3)$$

with the initial condition, $\widetilde{c_1}(t_0) = \widetilde{c_{01}}, \widetilde{c_2}(t_0) = \widetilde{c_{02}}$

IV. Phosphorus compartment model in fuzzy environment:

If we apply generalised Hukuhara derivative (Ghd) to the proposed fuzzy phosphorus model, then the following different cases arise as

(i) $\tilde{c_1}(t)$ and $\tilde{c_2}(t)$ both are (i)-gHD.
(ii) $\tilde{c_1}(t)$ is (i)-gHD and $\tilde{c_2}(t)$ is (ii)-gHD.
(iii) $\tilde{c_1}(t)$ is (ii)-gHD and $\tilde{c_2}(t)$ is (i)-gHD.
(iv) $\tilde{c_1}(t)$ and $\tilde{c_2}(t)$ both are (ii)-gHD

It is not necessary to display every case. Anyone can easily determine the outcomes of the other cases if he understands the methods used in any one of them. We only consider the following two instances for this paper:

i. Case 1: when $\widetilde{c_1}(t)$ and $\widetilde{c_2}(t)$ both are (i)-gHD:

The two compartment phosphorus model (3) converted into following system of FDE as follows

$$\frac{dc_{1x}(t,\alpha)}{dt} = \frac{W_{\tilde{\eta}}}{v_{\eta_1}} - \frac{Q_{\tilde{\eta}} c_{1y}(t,\alpha)}{v_{\eta_1}} - \frac{V_s A_{\tilde{\eta}} c_{1y}(t,\alpha)}{v_{\eta_1}} + \frac{K_s A_{\tilde{\eta}} c_{2x}(t,\alpha)}{v_{\eta_1}}$$
$$\frac{dc_{1y}(t,\alpha)}{dt} = \frac{W_{\tilde{\eta}}}{v_{\eta_1}} - \frac{Q_{\tilde{\eta}} c_{1x}(t,\alpha)}{v_{\eta_1}} - \frac{V_s A_{\tilde{\eta}} c_{1x}(t,\alpha)}{v_{\eta_1}} + \frac{K_s A_{\tilde{\eta}} c_{2y}(t,\alpha)}{v_{\eta_1}} \quad (4)$$
$$\frac{dc_{2x}(t,\alpha)}{dt} = \frac{V_s A_{\tilde{\eta}} c_{1x}(t,\alpha)}{v_{\eta_2}} - \frac{K_s A_{\tilde{\eta}} c_{2y}(t,\alpha)}{v_{\eta_2}} - \frac{K_b A_{\tilde{\eta}} c_{2y}(t,\alpha)}{v_{\eta_2}}$$
$$\frac{dc_{2y}(t,\alpha)}{dt} = \frac{V_s A_{\tilde{\eta}} c_{1y}(t,\alpha)}{v_{\eta_2}} - \frac{K_s A_{\tilde{\eta}} c_{2x}(t,\alpha)}{v_{\eta_2}} - \frac{K_b A_{\tilde{\eta}} c_{2x}(t,\alpha)}{v_{\eta_2}}$$

with the initial condition,

$c_{1x}(t_0,\alpha) = c_{01x}(\alpha), c_{1y}(t_0,\alpha) = c_{01y}(\alpha), c_{2x}(t_0,\alpha) = c_{02x}(\alpha), c_{2y}(t_0,\alpha) = c_{02y}(\alpha)$

The α -cut of $\widetilde{c_1}(t)$ and $\widetilde{c_2}(t)$,

$\tilde{c_1}(t,\alpha) =< c_{1x}(t,\alpha), c_{1y}(t,\alpha) >, \tilde{c_2}(t,\alpha) =< c_{2x}(t,\alpha), c_{2y}(t,\alpha) >$

1. Existence of equilibrium point:

The converted model (4) have coexistence of equilibrium point is

$E^*_{U1}(c^*_{1x}(t,\alpha), c^*_{1y}(t,\alpha), c^*_{2x}(t,\alpha), c^*_{2y}(t,\alpha))$ where, $c^*_{1x}(t,\alpha) = c^*_{1y}(t,\alpha) = \frac{W_{\tilde{\eta}}(K_b+K_s)}{Q_{\tilde{\eta}}(K_b+K_s)+V_s K_b A_{\tilde{\eta}}}$, and

$c^*_{2x}(t,\alpha) = c^*_{2y}(t,\alpha) = \frac{W_{\tilde{\eta}} V_s}{Q_{\tilde{\eta}}(K_b+K_s)+V_s K_b A_{\tilde{\eta}}}$, $(K_b + K_s) > 0, V_s > 0, K_b > 0$.

2. Stability Analysis:

Theorem 1: The transformed system (4) is unstable at E^*_{U1}.

Proof: The V- matrix at E^*_{U1} is given by,

$$V^*_{1p} = \begin{pmatrix} 0 & -p_{\gamma 1} & p_{\gamma 2} & 0 \\ -p_{\gamma 1} & 0 & 0 & p_{\gamma 2} \\ p_{\gamma 3} & 0 & 0 & -p_{\gamma 4} \\ 0 & p_{\gamma 3} & -p_{\gamma 4} & 0 \end{pmatrix}$$

where, $p_{\gamma 1} = \left(\frac{Q_{\tilde{\eta}}}{v_{\eta_1}} + \frac{V_s A_{\tilde{\eta}}}{v_{\eta_1}}\right)$, $p_{\gamma 2} = \frac{V_s A_{\tilde{\eta}}}{v_{\eta_2}}, p_{\gamma 3} = \frac{K_s A_{\tilde{\eta}}}{v_{\eta_1}}, p_{\gamma 4} = \frac{(K_s+K_b)A_{\tilde{\eta}}}{v_{\eta_2}}$

If μ_ξ be the eigen value of V^*_{1p}, the characteristic equation of V^*_{1p} is,

$$\mu_\xi^4 - (p_{\gamma 1}^2 + p_{\gamma 4}^2 + 2p_{\gamma 2}p_{\gamma 3})\mu_\xi^2 - p_{\gamma 1}p_{\gamma 2}p_{\gamma 3}p_{\gamma 4} - p_{\gamma 2}^2 p_{\gamma 3}^2 = 0 \quad (5)$$

From the above equation, the coeffi. of μ_ξ^3, μ_ξ will be zero. Therefore, converted model (4) is unstable at E^*_{U1}.

ii. Case 2: when $\widetilde{c_1}(t)$ and $\widetilde{c_2}(t)$ both are (ii)-gHD:

The two compartment phosphorus model (3) converted into following system of FDE as follows

$$\frac{dc_{1x}(t,\alpha)}{dt} = \frac{W_{\tilde{\eta}}}{v_{\eta_1}} - \frac{Q c_{1x}(t,\alpha)}{v_{\eta_1}} - \frac{V_s A_{\tilde{\eta}} c_{1x}(t,\alpha)}{v_{\eta_1}} + \frac{K_s A_{\tilde{\eta}} c_{2y}(t,\alpha)}{v_{\eta_1}}$$
$$\frac{dc_{1y}(t,\alpha)}{dt} = \frac{W_{\tilde{\eta}}}{v_{\eta_1}} - \frac{Q c_{1y}(t,\alpha)}{v_{\eta_1}} - \frac{V_s A_{\tilde{\eta}} c_{1y}(t,\alpha)}{v_{\eta_1}} + \frac{K_s A_{\tilde{\eta}} c_{2x}(t,\alpha)}{v_{\eta_1}} \quad (6)$$
$$\frac{dc_{2x}(t)}{dt} = \frac{V_s A_{\tilde{\eta}} c_{1y}(t)}{v_{\eta_2}} - \frac{K_s A_{\tilde{\eta}} c_{2x}(t)}{v_{\eta_2}} - \frac{K_b A_{\tilde{\eta}} c_{2x}(t)}{v_{\eta_2}}$$
$$\frac{dc_{2y}(t)}{dt} = \frac{V_s A_{\tilde{\eta}} c_{1x}(t)}{v_{\eta_2}} - \frac{K_s A_{\tilde{\eta}} c_{2y}(t)}{v_{\eta_2}} - \frac{K_b A_{\tilde{\eta}} c_{2y}(t)}{v_{\eta_2}}$$

with the initial state,
$c_{1x}(t_0,\alpha) = c_{01x}(\alpha), c_{1y}(t_0,\alpha) = c_{01y}(\alpha), c_{2x}(t_0,\alpha) = c_{02x}(\alpha), c_{2y}(t_0,\alpha) = c_{02y}(\alpha)$

The α -cut of $\tilde{c_1}(t)$ and $\tilde{c_2}(t)$ is given by,
$\tilde{c_1}(t,\alpha) =< c_{1x}(t,\alpha), c_{1y}(t,\alpha) >, \tilde{c_2}(t,\alpha) =< c_{2x}(t,\alpha), c_{2y}(t,\alpha) >$

1. Existence of equilibrium point:

The COE of transformed model (6) is given by,
$E^*_{U2}(c^*_{1x}(t,\alpha), c^*_{1y}(t,\alpha), c^*_{2x}(t,\alpha), c^*_{2y}(t,\alpha))$ where, $c^*_{1x}(t,\alpha) = c^*_{1y}(t,\alpha) = \frac{W_{\eta}(K_b+K_s)}{Q_{\eta}(K_b+K_s)+V_sK_bA_{\eta}}$, and
$c^*_{2x}(t,\alpha) = c^*_{2y}(t,\alpha) = \frac{W_{\eta}V_s}{Q_{\eta}(K_b+K_s)+V_sK_bA_{\eta}}$, $(K_b + K_s) > 0, V_s > 0, K_b > 0$.

2. Stability analysis:

Theorem 2: The converted system (6) is stable at E^*_{U2} when $2(q_{\gamma4} + q_{\gamma1})(q^2_{\gamma4} + q^2_{\gamma1} + 4q_{\gamma1}q_{\gamma4})(q_{\gamma1}q^2_{\gamma4} + q^2_{\gamma1}q_{\gamma4}) > (q_{\gamma4} + q_{\gamma1})^2(q^2_{\gamma1}q^2_{\gamma4} + q^2_{\gamma2}q^2_{13}) + 2(q_{\gamma1}q^2_{14} + q_{\gamma4}q^2_{\gamma1})^2$

Proof: The V-matrix at E^*_{U2} is given by,

$$V^*_{2p} = \begin{pmatrix} -q_{\gamma1} & 0 & 0 & q_{\gamma2} \\ 0 & -q_{\gamma1} & q_{\gamma2} & 0 \\ 0 & q_{\gamma3} & -q_{\gamma4} & 0 \\ q_{\gamma3} & 0 & 0 & -q_{\gamma4} \end{pmatrix}$$

where, $q_{\gamma1} = \left(\frac{Q}{v_{\eta_1}} + \frac{V_sA_{\eta}}{v_{\eta_1}}\right)$, $q_{\gamma2} = \frac{V_sA_{\eta}}{v_{\eta_2}}, q_{\gamma3} = \frac{K_sA_{\eta}}{v_{\eta_1}}, q_{\gamma4} = \frac{(K_s+K_b)A_{\eta}}{v_{\eta_2}}$
If μ_{m2} be the eigen value of V^*_{2p}, the characteristic eqn. of V^*_{2p} ,
$\mu^4_{m2} + A_{e1}\mu^3_{m2} + A_{e2}\mu^2_{m2} + A_{e3}\mu_{m2} + A_{e4} = 0$ (7)
where, $A_{e1} = 2(q_{\gamma4} + q_{\gamma1}), A_{e2} = (q^2_{\gamma4} + q^2_{\gamma1} + 4q_{\gamma1}q_{\gamma4}), A_{e3} = 2(q_{\gamma1}q^2_{\gamma4} + q_{\gamma4}q^2_{\gamma1}), A_{e4} = (q^2_{\gamma1}q^2_{\gamma4} + q^2_{\gamma2}q^2_{\gamma3})$
Using Routh-Hurwitz criteria for fourth degree polynomials,

(i) All coefficients are positive, where $A_{ei} > 0$, for $i = 1,2,3,4$.
(ii) $A_{e1}A_{e2} - A_{e3} > 0$ if $q^3_{\gamma4} + q^3_{\gamma1} + 4q_{\gamma1}q_{\gamma4}(q_{\gamma1} + q_{\gamma4}) > 0$.
(iii) $A_{e1}A_{e2}A_{e3} - A^2_{e1}A_{e4} - A^2_{e3} > 0$ if $2(q_{\gamma4} + q_{\gamma1})(q^2_{\gamma4} + q^2_{\gamma1} + 4q_{\gamma1}q_{\gamma4})(q_{\gamma1}q^2_{\gamma4} + q^2_{\gamma1}q_{\gamma4}) > (q_{\gamma4} + q_{\gamma1})^2(q^2_{\gamma1}q^2_{\gamma4} + q^2_{\gamma2}q^2_{13}) + 2(q_{\gamma1}q^2_{14} + q_{\gamma4}q^2_{\gamma1})^2$

1 Numerical simulation

In this section, we explored the system (4) and (6) rigorous numerical simulations to check and confirm all theoretical calculations using the MATLAB software. We investigated the dynamical behaviour of a two-compartment phosphorus model (2) in the presence of a fuzzy environment. To test the influence of fuzzy parameters in the proposed model (2), use the initial condition triangular fuzzy number (TFN) and set all of the parameter values indicated in cases 1 and 2 as follows

Case 1: When both $\tilde{c_1}(t)$ and $\tilde{c_2}(t)$ possess (i)-gHD

In this section we considering all the parameter value used in the proposed system (4) are reported in Table 1 and rate of two concentration of phosphorus in water at t=0 i.e initial condition of $\tilde{c}_1(t_0)$ and $\tilde{c}_2(t_0)$ of the system (4) is taken TFN as
$c_{1x}(t_0,\alpha) = 40 + 2\alpha, c_{1y}(t_0,\alpha) = 45 - 3\alpha, c_{2x}(t_0,\alpha) = 30 + 2\alpha, c_{2y}(t_0,\alpha) = 35 - 3\alpha$ (7)
For the transformed model (4) we set the parameter as follows,

Table 1: The parameter values

Parameters	A_{η}	V_s	K_b	K_s	W_{η}	Q_{η}	v_{η_1}	v_{η_2}
Value	1.5×10^3	0.0005	8.03×10^{-4}	0.0031	11410	1928	53×10^6	8×10^3
Source	[48]	[48]	[49]	assumed	[48]	[49]	[48]	[48]

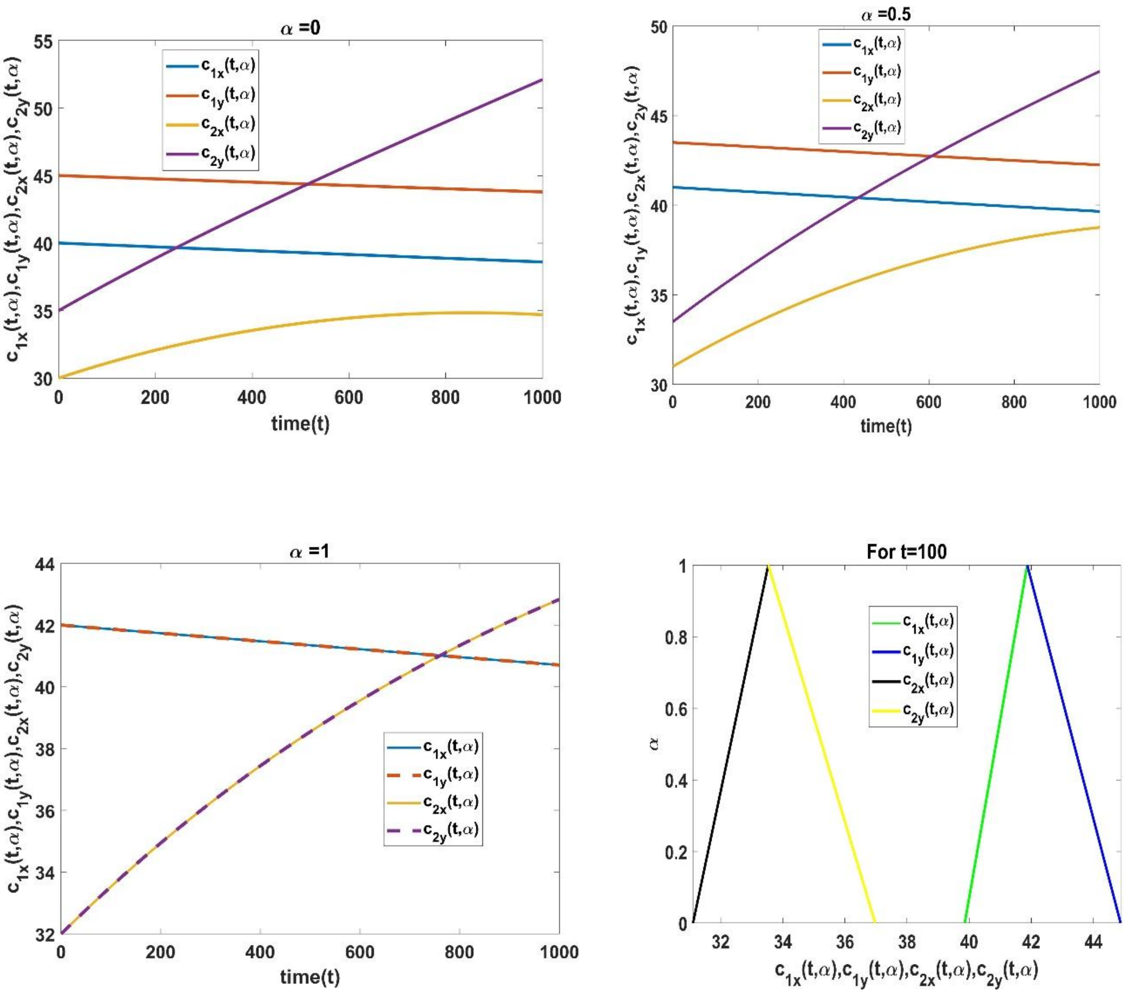

Figure 1**:** Time series solution of the system (4) in Figure 1(a) for α=0, in Figure 1(b) for α=0.5, in Figure 1(c) for α=1, for $t \in [0,1000]$. In Figure 1(d), depicts the membership function of $c_{1x}(t,\alpha), c_{1y}(t,\alpha)$, $c_{2x}(t,\alpha), c_{2y}(t,\alpha)$ vs. α for $t = 100$ represents triangular fuzzy number.

Table 2: Fuzzy solution of $c_{1x}(t,\alpha), c_{1y}(t,\alpha), c_{2x}(t,\alpha), c_{2y}(t,\alpha)$ of the (4) for $t = 100$.

α	$c_{1x}(t,\alpha)$	$c_{1y}(t,\alpha)$	$c_{2x}(t,\alpha)$	$c_{2y}(t,\alpha)$
0	39.8577	44.8760	31.1091	36.9760
0.1	40.0588	44.5753	31.3515	36.6316
0.2	40.2599	44.2746	31.5938	36.2873
0.3	40.4610	43.9738	31.8362	35.9430
0.4	40.6621	43.6731	32.0785	35.5987
0.5	40.8632	43.3724	32.3209	35.2543
0.6	41.0643	43.0716	32.5633	34.9100
0.7	41.2654	42.7709	32.8056	34.5657
0.8	41.4665	42.4701	33.0480	34.2213
0.9	41.6676	42.1694	33.2903	33.8770
1	41.8687	41.8687	33.5327	33.5327

In table 2 reflects $c_{1x}(t,\alpha)$ is increasing, $c_{1y}(t,\alpha)$ is decreasing; $c_{2x}(\text{t},\alpha)$ is increasing, $c_{2y}(t,\alpha)$is decreasing for $\alpha \in [0,1]$ at $t = 100$. Hence, $\tilde{c}_1(t), \tilde{c}_1(t)$ gives the strong fuzzy solution of the system (4).
Using the values of Table 1, Table 2 and the above initial condition of α −cut of $\tilde{c}_1(t), \tilde{c}_2(t)$, draw the Figure 1(a),1(b) and 1(c) for α=0,0.5,1 respectively. Figure 1 shows the time series solution for system (4) in a fuzzy environment. We can observe that $c_{1x}(t,\alpha) \leq c_{1y}(t,\alpha)$, $c_{2x}(t,\alpha) \leq c_{2y}(t,\alpha)$ when $t \in [0,1000]$ in Figures 1(a), 1(b), and 1(c). Therefore, at $t = 100$, every solution of system (4) confirms a fuzzy solution where $t \in [0,\alpha]$. The instability of system (4) at the coexistence equilibrium point E_1^* is evident from Figure 1.

Case 2: When both $\tilde{c_1}(t)$ and $\tilde{c_2}(t)$ possess (ii)-gHD

Using the same values of $A_{\tilde{\eta}}, V_s, K_s, K_b, W_{\tilde{\eta}}, Q, v_{\eta_1}, v_{\eta_2}$ from case 1 and consider same initial condition of α-cut are reported in (7). The values of $c_{1x}(t,\alpha), c_{1y}(t,\alpha), c_{2x}(t,\alpha), c_{2y}(t,\alpha)$ are tabulated in Table 3 for α∈ [0,1] at $t = 200$.

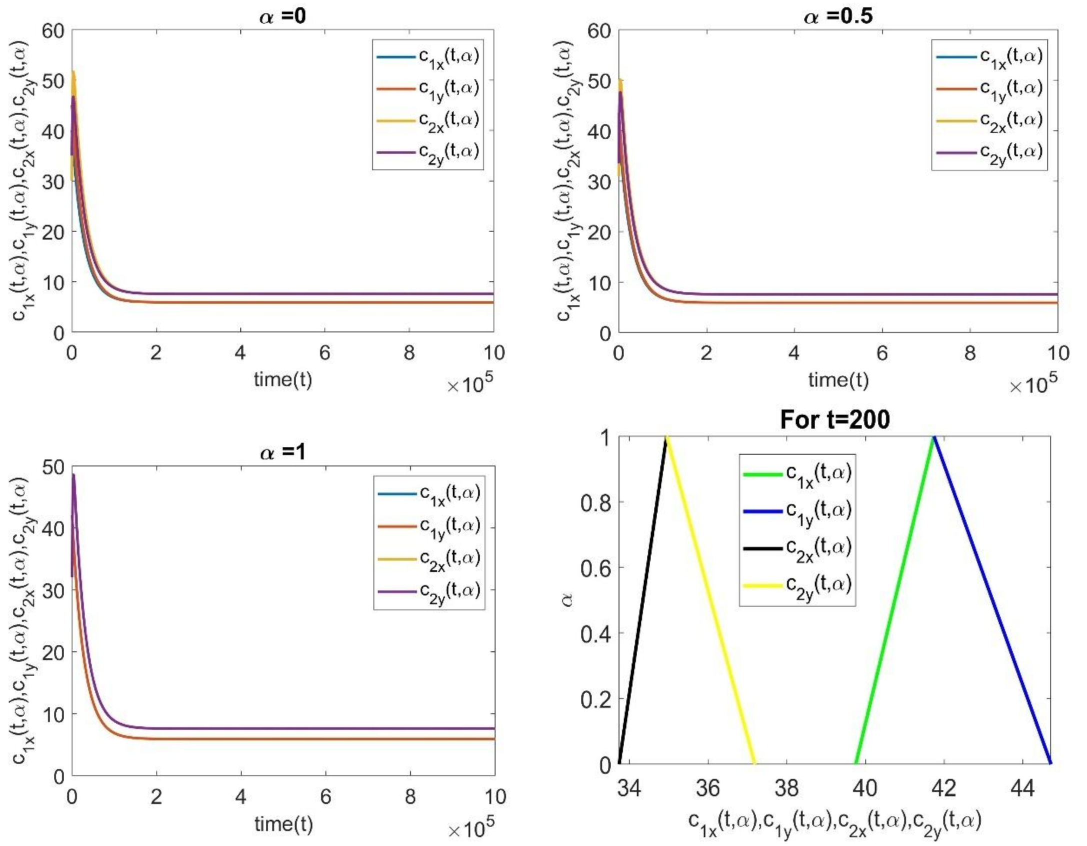

Figure 2: Time series solution of the system (6) in Figure 1(a) for α=0, in Figure 1(b) for α=0.5, in Figure 1(c) for α=1, for $t \in [0, 10^5]$. In Figure 1(d), depicts the membership function of $c_{1x}(t,\alpha), c_{1y}(t,\alpha)$, $c_{2x}(t,\alpha), c_{2y}(t,\alpha)$ vs. α for $t = 200$ represents triangular fuzzy number.

Table 3: Fuzzy solution of $c_{1x}(t,\alpha), c_{1y}(t,\alpha), c_{2x}(t,\alpha), c_{2y}(t,\alpha)$ of the (6) for $t = 200$.

α	$c_{1x}(t,\alpha)$	$c_{1y}(t,\alpha)$	$c_{2x}(t,\alpha)$	$c_{2y}(t,\alpha)$
0	39.7524	44.7160	33.7390	37.1894
0.1	39.9510	44.4182	33.8597	36.9650
0.2	40.1495	44.1204	33.9803	36.7406
0.3	40.3481	43.8225	34.1009	36.5162
0.4	40.5466	43.5247	34.2216	36.2918
0.5	40.7451	43.2269	34.3422	36.0674
0.6	40.9437	42.9291	34.4628	35.8430
0.7	41.1422	42.6313	34.5835	35.6186
0.8	41.3408	42.3335	34.7041	35.3942
0.9	41.5393	42.0357	34.8248	35.1698
1	41.7378	41.7378	34.9454	34.9454

Table 3 shows that $c_{1x}(t,\alpha)$ is increasing, $c_{1y}(t,\alpha)$ is decreasing, $c_{2x}(t,\alpha)$ is increasing, and $c_{2y}(t,\alpha)$ is decreasing for $\alpha \in [0,1]$ at $t = 200$. Thus, $\tilde{c}_1(t), \tilde{c}_1(t)$ provides a strong fuzzy solution for the problem (6). Figures 2(a), 2(b), and 2(c) are drawn using the initial condition of the α-cut of $\tilde{c}_1(t), \tilde{c}_2(t)$, and based on the values in Tables 1 and 3. Figure 2 shows the time series solution for the system (6) in a fuzzy environment. Figures 2(a), 2(b), and 2(c) show that $c_{1x}(t,\alpha) \leq c_{1y}(t,\alpha)$, $c_{2x}(t,\alpha) \leq c_{2y}(t,\alpha)$ where $t \in [0, 10^5]$.This implies that the system (6) is stable at the coexistence equilibrium point E_2^*.

V. Result and Discussion

To demonstrate the entire work which is presented in the following way

S1: Start the phosphorus in lake water model with parameter $W_{\tilde{\eta}}$ (=11410), $Q_{\tilde{\eta}}$ (=1928),$V_s(= 0.0005)$,$K_s(= 0.0031)$,$K_b(= 8.03 \times 10^{-4})$, $A_{\tilde{\eta}}$ (=1.5×10^3),$v_{\eta_1}(= 53 \times 10^6)$, and$v_{\eta_2}(= 8 \times 10^3)$. [47]

S2: Develop the suggested model assuming the initial conditions are represented by fuzzy numbers.

S3: Use Hukuhara differentiability to convert fuzzy phosphorus in lake water model into a system of differential equations with its parametric form.

S4: Obtain the existence of equilibrium points and discuss the stability criteria for each case.

S5: Perform all necessary computations and validate the results using numerical methods.

S6: Verify all fuzzy solution are strong /weak TFN solution with respective input.

S7: Stop.

VI. Conclusion

In recent times, several forms of biological modeling have relied heavily on mathematical modeling. In this study, we have presented a model of phosphorus concentrations in lake water and sediment using fuzzy set theory concepts whereas all parameters are treated as TFN. To determine the fuzzy solution of the suggested model, we have used the idea of generalized Hukuhara fuzzy derivative techniques. By the help of (i)gH and (ii) gH fuzzy differentiability approach the fuzzy phosphorus in lake water model is transformed into a system of differential equations with α-cut form. Equilibrium points and stability criteria of suggested model have been manifested in fuzzy environment. At time points 100 and 200, the values of the solutions for both systems follow the membership function of a triangular fuzzy number (refer to Tables 2 and 3, respectively). We have shown that all fuzzy solution are strong fuzzy solution.All numerical results and graphs have been verified by Matlab software. One interesting area of research is the phosphorus in lake water model. The study of the phosphorus in lake water model employing two or more first-order systems of equations in a neutrosophic environment can be expanded upon by this work. Delay differential equations (DDEs) are another modelling approach. They capture time delays in the system, which can be crucial for the phosphorus in lake water model. In future, delay phosphorus in lake water model may be discussed via neutrosophic differential equation.

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