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An Adaptive Mixed Finite Element Method for Second Order Elliptic Eigenvalue Problems

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ABSTRACT: *This paper studies the Raviart-Thomas mixed finite element method for solving the general second-order elliptic eigenvalue problem. It derives and analyzes a class of stable residual-type posterior error estimators, and through theoretical analysis, proves the effectiveness and reliability of the proposed posterior error indicators. Based on the posterior error estimate in this paper, we develop an adaptive algorithm to solve the second-order elliptic eigenvalue problem, and the numerical results demonstrate that the adaptive algorithm established in this paper is efficient.*

KEYWORDS - *Second-order elliptic eigenvalue problem, a posterior error, mixed finite element, adaptive algorithm.*

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I. INTRODUCTION

The second-order elliptic eigenvalue problem is a classical and widely applied topic in mathematical physics, with significant importance in fields such as engineering, physics, and computational science. Reference [1] discusses extrapolation methods for eigenvalue problems, reference [2] explores reconstruction algorithms for function values, and reference [3] addresses multi-level correction methods for eigenvalue problems, among others. The mixed finite element method, as an important branch of finite element methods, has gained increasing attention in practical applications. Unlike traditional standard finite element methods, the mixed finite element method enhances the mathematical expressiveness of the model by simultaneously considering multiple physical quantities (such as displacement, stress, pressure, etc.) as unknowns, and introducing additional variables and constraints. This method provides higher solution accuracy and stronger numerical stability when handling complex problems.

With the continuous development of computer technology and numerical methods, the proposal and application of adaptive methods have become one of the important advancements in finite element analysis. In 1978, Babuska and Rheinboldt [4][5] first introduced adaptive methods and residual-based a posteriori error estimation, which attracted widespread attention in the field of scientific and engineering computation, leading to significant achievements. In recent years, this method has become one of the mainstream approaches in scientific computing and has been extensively researched (see [6][7]), with reference [8] discussing the application of posterior error estimation and adaptive algorithms.

This paper primarily uses the mixed finite element method to solve the second-order elliptic eigenvalue problem, proposing a residual-based a posteriori error indicator and verifying its reliability and effectiveness. Based on this, adaptive computation is implemented. Numerical results show that the adaptive algorithm achieves optimal convergence rates. Furthermore, the error curves indicate that, for the same degrees of freedom, the approximation obtained using the adaptive algorithm is more accurate than that obtained using the uniform mesh method.

1.1 Notations and Basic Preparation:

The following is a description of the notation that will be used in this article. For $s > 0$, we denote as $\|\cdot\|_{s,\Omega}$ the norms of the Sobolev space $H^s(\Omega)$ and $[H^s(\Omega)]^2$, with the convention $H^0(\Omega) = L^2(\Omega)$ and $[H^0(\Omega)]^2 = [L^2(\Omega)]^2$. In addition, we define the Hilbert space as follows

$$H(div,\Omega) = \left\{\boldsymbol{\tau} \in \left(L^2(\Omega)\right)^2 : div\boldsymbol{\tau} \in L^2(\Omega)\right\}$$

and the corresponding norm is given by:

$$\|\boldsymbol{\tau}\|_{H(div,\Omega)}^2 = \|\boldsymbol{\tau}\|_{0,\Omega}^2 + \|div\boldsymbol{\tau}\|_{0,\Omega}^2 \tag{1.1}$$

The Poincaré inequality: If Ω is a connected and bounded convex domain in one direction, then for any $v \in H^1(\Omega)$, we have:

$$\|v\|_{0,\Omega} \lesssim \|\nabla v\|_{0,\Omega} \tag{1.2}$$

The definition of the curl is as follows:

$$\boldsymbol{curl}\phi = \left(\frac{\partial\phi}{\partial x_2}, -\frac{\partial\phi}{\partial x_1}\right),\ rot\boldsymbol{\varphi} = \frac{\partial\varphi_2}{\partial x_1} - \frac{\partial\varphi_1}{\partial x_2} \tag{1.3}$$

Finally, the relation $a \lesssim b$ represents $a \leq Cb$, where C denotes a constant independent of h, the mesh size, and similarly, $a \gtrsim b$ represents $a \geq Cb$.

II. THE SECOND-ORDER ELLIPTIC EIGENVALUE PROBLEM

Consider the second-order elliptic eigenvalue problem: Find $\lambda \in R$, $u \in H_0^1(\Omega)$, such that

$$\begin{cases} -\nabla \cdot (c(x)\nabla u) = \lambda u, & in \quad \Omega \\ u = 0, & on \quad \partial\Omega \end{cases} \tag{2.1}$$

where $c(x) \geq c_0 > 0$, $\Omega \subset R^2$ is a bounded domain with a Lipschitz boundary, and ∇ , $\nabla \cdot$ denote the gradient operator and the divergence operator, respectively.

Let $\boldsymbol{\sigma} = c(x) \cdot \nabla u$, then the problem (2.1) is equivalent to

$$\begin{cases} c(x)^{-1}\boldsymbol{\sigma} - \nabla u = 0, & in \quad \Omega \\ -div\boldsymbol{\sigma} = \lambda u, & in \quad \Omega \\ u = 0, & on \quad \partial\Omega \end{cases} \tag{2.2}$$

let $\boldsymbol{H} = H(div, \Omega)$, $V = G = L^2(\Omega)$, from the equivalent form (2.2), the mixed variational form of the problem (2.1) is obtained as follows: Find $(\lambda, \boldsymbol{\sigma}, u) \in R \times \boldsymbol{H} \times V$, such that

$$\begin{cases} a(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, u) = 0, & \forall \boldsymbol{\tau} \in \boldsymbol{H} \\ b(\boldsymbol{\sigma}, v) = -\lambda r(u, v), & \forall v \in V \end{cases} \tag{2.3}$$

where the bilinear forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$, and $r(\cdot,\cdot)$ are defined as follows:

$$a(\boldsymbol{\sigma}, \boldsymbol{\tau}) = \int_\Omega c^{-1}\boldsymbol{\sigma\tau}dx, \quad b(\boldsymbol{\tau}, v) = \int_\Omega div\boldsymbol{\tau} \cdot vdx, \quad r(u, v) = \int_\Omega uvdx$$

and the bilinear forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$, and $r(\cdot,\cdot)$ have the following properties

$$|a(\boldsymbol{\sigma}, \boldsymbol{\tau})| \lesssim \|\boldsymbol{\sigma}\|_{\boldsymbol{H}}\|\boldsymbol{\tau}\|_{\boldsymbol{H}}, \ \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \boldsymbol{H} \tag{2.4}$$

$$a(\boldsymbol{\sigma}, \boldsymbol{\sigma}) \gtrsim \|\boldsymbol{\sigma}\|_{\boldsymbol{H}}^2, \quad \forall \boldsymbol{\sigma} \in \boldsymbol{H} \tag{2.5}$$

$$|b(\boldsymbol{\tau}, v)| \lesssim \|\tau\|_{\boldsymbol{H}}\|v\|_V, \ \forall \boldsymbol{\tau} \in \boldsymbol{H}, v \in V \tag{2.6}$$

$$|r(u, v)| \lesssim \|u\|_V\|v\|_V, \ \forall u, v \in V \tag{2.7}$$

For the eigenvalue λ, there exists the Rayleigh quotient expression

$$\lambda = \frac{a(\boldsymbol{\sigma}, \boldsymbol{\sigma})}{r(u,u)} \tag{2.8}$$

From [9][12], the eigenvalue problem (2.3) has an eigenvalue sequence $\{\lambda_j\}$

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots, \qquad \lim_{k\to\infty}\lambda_k = \infty$$

and the associated eigenfunctions

$$(\boldsymbol{\sigma}_1, u_1), (\boldsymbol{\sigma}_2, u_2), \cdots, (\boldsymbol{\sigma}_k, u_k), \cdots$$

III. STANDARD MIXED FINITE ELEMENT APPROXIMATION

Let $\mathcal{T}_h = \{\kappa\}$ be a shape-regular mesh of Ω, where h_κ denotes the diameter of each element κ, and $h = \max_{\kappa\in\mathcal{T}_h} h_\kappa$. Let $\mathcal{E}_h$ be the collection of all edges, with h_e representing the length of edge $e \in \mathcal{E}_h$. For any $\kappa \in \mathcal{T}_h$, we denote by $\mathcal{P}_k(\kappa)$ the space of polynomials defined on element κ, where $k \geq 0$. With these ingredients at hand, we define the Raviart-Thomas space as follows (see [10])

$$\boldsymbol{H}_h = \{\boldsymbol{\tau}_h \in H(div, \Omega): \boldsymbol{\tau}_h|_\kappa \in [\mathcal{P}_k(\kappa)]^2 \oplus \boldsymbol{x} \cdot \mathcal{P}_k(\kappa) \quad \forall \kappa \in \mathcal{T}_h\} \tag{3.1}$$

$$V_h = \{v_h \in L^2(\Omega): v_h|_\kappa \in \mathcal{P}_k(\kappa) \quad \forall \kappa \in \mathcal{T}_h\} \tag{3.2}$$

Then, according to the definitions of spaces $\boldsymbol{H}_h$ and V_h, we have

$$div\boldsymbol{H}_h = V_h \tag{3.3}$$

The mixed finite element approximation of problem (2.3) is: Find $(\lambda_h, \boldsymbol{\sigma}_h, u_h) \in R \times \boldsymbol{H}_h \times V_h$, such that

$$\begin{cases} a(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h, u_h) = 0, & \forall \boldsymbol{\tau}_h \in \boldsymbol{H}_h \\ b(\boldsymbol{\sigma}_h, v_h) = -\lambda_h r(u_h, v_h), & \forall v_h \in V_h \end{cases} \tag{3.4}$$

for the eigenvalue λ_h, there exists the Rayleigh quotient expression

$$\lambda_h = \frac{a(\sigma_h, \sigma_h)}{r(u_h, u_h)} \tag{3.5}$$

From [9],[12], the eigenvalue problem (3.4) has eigenvalues as follow

$$0 \le \lambda_{1,h} \le \lambda_{2,h} \le \cdots \le \lambda_{k,h} \le \cdots \le \lambda_{N,h}, \lim_{k\to\infty} \lambda_k = \infty$$

and the associated eigenfunctions

$$(\boldsymbol{\sigma}_{1,h}, u_{1,h}), (\boldsymbol{\sigma}_{2,h}, u_{2,h}), \cdots, (\boldsymbol{\sigma}_{k,h}, u_{k,h}), \cdots, (\boldsymbol{\sigma}_{N,h}, u_{N,h})$$

Define the L^2- projection operator $\Pi_h: L^2(\Omega) \to V_h$, which satisfies

$$\|v - \Pi_h v\|_0 \lesssim h|v|_1 \quad \forall v \in H^1(\Omega) \tag{3.6}$$

Define the Raviart-Thomas interpolation operator $\boldsymbol{Q}_h: \boldsymbol{W} \to \boldsymbol{H}_h$, where $\boldsymbol{W} = H(div, \Omega) \cap [L^s(\Omega)]^2$, for $s > 2$, it satisfies

$$\|\boldsymbol{\sigma} - \boldsymbol{Q}_h \boldsymbol{\sigma}\|_0 \lesssim h|\boldsymbol{\sigma}|_1, \ \forall \boldsymbol{\sigma} \in H^1(\Omega) \cap H(div, \Omega) \tag{3.7}$$

therefore, the projection operator Π_h and the interpolation operator $\boldsymbol{Q}_h$ satisfy the following commutative property

$$div \boldsymbol{Q}_h = \Pi_h div \tag{3.8}$$

Let Id denote the identity, and $\perp$ denote the L^2-orthogonality, then we have

$$div(Id - \boldsymbol{Q}_h)\boldsymbol{W} \perp V_h \tag{3.9}$$

Furthermore, we assume that the interpolation satisfies a local error estimate, then we have

$$\|h^{-1}(Id - \boldsymbol{Q}_h)\boldsymbol{\tau}\|_0 \lesssim |\boldsymbol{\tau}|_{1,\cup\mathcal{T}_h}, \boldsymbol{\tau} \in H^1(\Omega) \cap H(div, \Omega) \tag{3.10}$$

Finally, assuming that the interpolation operator $\boldsymbol{Q}_h$ approximates the normal component on the boundary, then for any $e \in \mathcal{E}_h$, $v_h \in V_h$, and $\boldsymbol{\tau} \in \boldsymbol{W}$, we can obtain

$$\int_\kappa v_h \cdot (Id - \boldsymbol{Q}_h)\boldsymbol{\tau} \cdot \boldsymbol{n} dx = 0 \tag{3.11}$$

where $\boldsymbol{n} = (n_1, n_2)$ denote the unit outward normal vector.

Consider the source problem corresponding to the eigenvalue problem (2.3) and its discrete mixed finite element form.

Find $(\boldsymbol{p}, q) \in \boldsymbol{H} \times V$, such that

$$\begin{cases} a(\boldsymbol{p}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, q) = 0, & \forall \boldsymbol{\tau} \in \boldsymbol{H} \\ b(\boldsymbol{p}, v) = -(f, v), & \forall v \in V \end{cases} \tag{3.12}$$

Find $(\boldsymbol{p}_h, q_h) \in \boldsymbol{H}_h \times V_h$, such that

$$\begin{cases} a(\boldsymbol{p}_h, \boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h, q_h) = 0, & \forall \boldsymbol{\tau}_h \in \boldsymbol{H}_h \\ b(\boldsymbol{p}_h, v_h) = -(f, v_h), & \forall v_h \in V_h \end{cases} \tag{3.13}$$

For the polygonal domain, it is known from [11] that the problem (3.12) has a unique solution, and the following regularity result holds: For $f \in V$, $\forall(\boldsymbol{p}, q) \in H^r(\Omega)^2 \times [H^{1+r}(\Omega) \cap H_0^{1+r}(\Omega)]$, such that

$$\|q\|_{1+r} + \|\boldsymbol{p}\|_r \lesssim \|f\|_0, \quad 1/2 < r \le 1 \tag{3.14}$$

Assume that the mixed finite element spaces $\boldsymbol{H}_h \subset \boldsymbol{H}$ and $V_h \subset V$ satisfy the inf-sup condition, i.e., there exists a constant $\beta > 0$ such that

$$\sup_{\boldsymbol{\tau}_h \in \boldsymbol{H}_h} \frac{b(\tau_h, v_h)}{\|\tau_h\|_{\boldsymbol{H}}} \ge \beta \|v_h\|_0, \quad \forall v_h \in V_h \tag{3.15}$$

Then (3.13) also exists a unique solution $(\boldsymbol{p}_h, q_h) \in \boldsymbol{H}_h \times V_h$ and the following error estimate is valid (see [10][16])

$$\|\boldsymbol{p} - \boldsymbol{p}_h\|_{\boldsymbol{H}} + \|q - q_h\|_0 \lesssim \inf_{\boldsymbol{\tau}_h \in \boldsymbol{H}_h} \|\boldsymbol{p} - \boldsymbol{\tau}_h\|_{\boldsymbol{H}} + \inf_{v_h \in V_h} \|q - v_h\|_0 \tag{3.16}$$

Therefore, we define linear bounded operators

$$\boldsymbol{S}: G \to \boldsymbol{H} \subset G \qquad T: G \to V \subset G$$
$$\boldsymbol{S}_h: G \to \boldsymbol{H}_h \subset G \qquad T_h: G \to V_h \subset G$$

For $f \in V$, $\forall(\boldsymbol{S}f, Tf) \in \boldsymbol{H} \times V$ such that

$$\begin{cases} a(\boldsymbol{S}f, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, Tf) = 0, & \forall \boldsymbol{\tau} \in \boldsymbol{H} \\ b(\boldsymbol{S}f, v) = -(f, v), & \forall v \in V \end{cases} \tag{3.17}$$

For $f \in V$, $\forall(\boldsymbol{S}_h f, T_h f) \in \boldsymbol{H}_h \times V_h$ such that

$$\begin{cases} a(\boldsymbol{S}_h f, \boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h, T_h f) = 0, & \forall \boldsymbol{\tau}_h \in \boldsymbol{H}_h \\ b(\boldsymbol{S}_h f, v_h) = -(f, v_h), & \forall v_h \in V_h \end{cases} \tag{3.18}$$

Thus, the problems (2.3) and (3.4) have equivalent operator forms, respectively

$$\lambda T u = u, \;\; \boldsymbol{S}(\lambda u) = \boldsymbol{\sigma} \tag{3.19}$$

$$\lambda_h T_h u_h = u_h, \;\; \boldsymbol{S}_h(\lambda_h u_h) = \boldsymbol{\sigma_h} \tag{3.20}$$

It's easy to know T and T_h are self-adjoint. Suppose that λ and λ_h are the kth eigenvalue of (3.19) and (3.20), respectively, from (3.14), we can conclude that T is completely continuous.

The proof method similar to Theorem 3 in [12] can be used to prove the following lemma.

Lemma 3.1. Let $(\boldsymbol{p}, q)$ and $(\boldsymbol{p}_h, q_h)$ be the solutions of (3.12) and (3.13), respectively, then the following equation holds

$$\begin{aligned} \|q - q_h\|_0 = \sup_{d \in G, d \neq 0} \frac{1}{\|d\|_0} (b(\boldsymbol{\varrho} - \boldsymbol{Q}_h \boldsymbol{\varrho}, q - v_h) + a(\boldsymbol{p} - \boldsymbol{p}_h, \boldsymbol{\varrho} - \boldsymbol{Q}_h \boldsymbol{\varrho}) \\ + b(\boldsymbol{p} - \boldsymbol{Q}_h \boldsymbol{p}, \vartheta - y_h)), \qquad \forall v_h, y_h \in V_h \end{aligned} \tag{3.21}$$

where, for any $d \in G$, $(\boldsymbol{\varrho}, \vartheta) \in \boldsymbol{H} \times V$ is a general solution of (3.22):

$$\begin{cases} a(\boldsymbol{\varpi}, \boldsymbol{\varrho}) + b(\boldsymbol{\varpi}, \vartheta) = 0, & \forall \boldsymbol{\varpi} \in \boldsymbol{H} \\ b(\boldsymbol{\varrho}, v) = -(d, v), & \forall v \in V \end{cases} \tag{3.22}$$

For the above problem (3.22), there exists well-known regularity result $\|\boldsymbol{\varrho}\|_r + \|\vartheta\|_{1+r} \lesssim \|d\|_0$, where $1/2 < r \leq 1$.

Theorem 3.1. Under the conditions of Lemma 3.1, for any $f \in V$, the following estimates hold

$$\|q - q_h\|_0 \lesssim h^m \|q\|_m + h^{r+m} \|\boldsymbol{p}\|_m + h^{m+min(r,k)} \|div \boldsymbol{p}\|_{m-1}, \;\; 1 \leq m \leq k+1 \tag{3.23}$$

$$\|q - q_h\|_0 \lesssim h^{1+r} \|q\|_{1+r} + h^{1+min(r,k)} \|f\|_0 + h^{2r} \|f\|_0, \qquad 1/2 < r \leq 1. \tag{3.24}$$

Proof. First, we estimate the first term $b(\boldsymbol{\varrho} - \boldsymbol{Q}_h \boldsymbol{\varrho}, q - v_h)$ in the equation (3.21). There exists an operator $\Sigma_h : H^m(\Omega) \to V_h$ such that for $q \in H^m(\Omega)$, we have

$$\|q - \Sigma_h q\|_0 \lesssim h^m \|q\|_m, \quad 1 \leq m \leq k+1 \tag{3.25}$$

Therefore, using equation (3.25) and the definition of the interpolation operator $\boldsymbol{Q}_h$, we obtain the estimate as follow

$$\begin{aligned} \inf_{v_h \in V_h} |b(\boldsymbol{\varrho} - \boldsymbol{Q}_h \boldsymbol{\varrho}, q - v_h)| &\lesssim \|div(\boldsymbol{\varrho} - \boldsymbol{Q}_h \boldsymbol{\varrho})\|_0 \inf_{v_h \in V_h} \|q - v_h\|_0 \\ &\lesssim \|div \boldsymbol{\varrho}\|_0 \|q - \Sigma_h q\|_0 \\ &\lesssim \|div \boldsymbol{\varrho}\|_0 h^m \|q\|_m, \;\; 1 \leq m \leq k+1 \end{aligned} \tag{3.26}$$

Next, we estimate the second term $b(\boldsymbol{p} - \boldsymbol{Q}_h \boldsymbol{p}, \vartheta - y_h)$. Similar to equation (3.26), we have the following estimate for $1 \leq m \leq k+1$

$$\begin{aligned} \inf_{v_h \in V_h} |b(\boldsymbol{p} - \boldsymbol{Q}_h \boldsymbol{p}, \vartheta - y_h)| &\lesssim \|div(\boldsymbol{p} - \boldsymbol{Q}_h \boldsymbol{p})\|_0 \|\vartheta - \Sigma_h \vartheta\|_0 \\ &\lesssim h^{m-1} \|div \boldsymbol{p}\|_{m-1} h^{1+min(r,k)} \|\vartheta\|_{1+r} \end{aligned} \tag{3.27}$$

Finally, using the equation (2.15) in [12], the estimate for the last term $a(\boldsymbol{p} - \boldsymbol{p}_h, \boldsymbol{\varrho} - \boldsymbol{Q}_h \boldsymbol{\varrho})$ is as follows

$$\begin{aligned} |a(\boldsymbol{p} - \boldsymbol{p}_h, \boldsymbol{\varrho} - \boldsymbol{Q}_h \boldsymbol{\varrho})| &\lesssim \|\boldsymbol{p} - \boldsymbol{p}_h\|_0 \|\boldsymbol{\varrho} - \boldsymbol{Q}_h \boldsymbol{\varrho}\|_0 \\ &\lesssim \|\boldsymbol{p} - \boldsymbol{Q}_h \boldsymbol{p}\|_0 h^r \|\boldsymbol{\varrho}\|_r \end{aligned} \tag{3.28}$$

Based on the above, we use the regularity result from the auxiliary problem (3.22), and by combining equations (3.26) - (3.28), then we can obtain the following estimate

$$\begin{aligned} \|q - q_h\|_0 &= \sup_{d \in G} \frac{1}{\|d\|_0} (b(\boldsymbol{\varrho} - \boldsymbol{Q}_h \boldsymbol{\varrho}, q - v_h) + a(\boldsymbol{p} - \boldsymbol{p}_h, \boldsymbol{\varrho} - \boldsymbol{Q}_h \boldsymbol{\varrho}) + b(\boldsymbol{p} - \boldsymbol{Q}_h \boldsymbol{p}, \vartheta - y_h)) \\ &\lesssim \frac{1}{\|d\|_0} \left(h^m \|div \boldsymbol{\varrho}\|_0 \|q\|_m + \|\boldsymbol{p} - \boldsymbol{Q}_h \boldsymbol{p}\|_0 h^r \|\boldsymbol{\varrho}\|_r + h^{m+min(r,k)} \|div \boldsymbol{p}\|_{m-1} \|\vartheta\|_{1+r} \right) \\ &\lesssim \frac{1}{\|d\|_0} \left(h^m \|d\|_0 \|q\|_m + \|\boldsymbol{p} - \boldsymbol{Q}_h \boldsymbol{p}\|_0 h^r \|\boldsymbol{\varrho}\|_r + h^{m+min(r,k)} \|div \boldsymbol{p}\|_{m-1} \|\vartheta\|_{1+r} \right) \\ &\lesssim h^m \|q\|_m + h^{r+m} \|\boldsymbol{p}\|_m + h^{m+min(r,k)} \|div \boldsymbol{p}\|_{m-1}, \qquad 1 \leq m \leq k+1 \end{aligned} \tag{3.29}$$

Based on equation (3.29), we can also derive the following conclusion

$$\|q - q_h\|_0 \lesssim h^{1+r} \|q\|_{1+r} + h^{1+min(r,k)} \|f\|_0 + h^{2r} \|f\|_0$$

From Theorem 3.1 and [17], we know the following prior error estimates: For any $f \in V$, the following hold

$$\|\boldsymbol{S}f - \boldsymbol{S}_h f\|_0 \lesssim h^m \|Tf\|_{m+1}, \;\; 1 \leq m \leq k+1 \tag{3.30}$$

$$\|Tf - T_h f\|_0 \lesssim h^m \|Tf\|_{m+1}, \;\; 1 \leq m \leq k+1 \tag{3.31}$$

Therefore, from equations (3.30) and (3.31), the following convergence result holds (see [9],[19],[20]).

$$\|T - T_h\|_{\mathcal{L}(V,V)} \to 0 \quad \text{if} \quad h \to 0 \tag{3.32}$$

$$\|\boldsymbol{S}-\boldsymbol{S}_h\|_{\mathcal{L}(V,\boldsymbol{H})} \to 0 \quad \text{if} \quad h \to 0 \tag{3.33}$$

The results in equations (3.32) and (3.33) are equivalent to the convergence of eigenvalues and eigenfunctions. Therefore, using the abstract theory from [14][15], and a priori results in equations (3.30) and (3.31), we may draw the following conclusions.

Lemma 3.1. Let $(\lambda, \boldsymbol{\sigma}, u)$ and $(\lambda_h, \boldsymbol{\sigma}_h, u_h)$ be the solutions to the eigenvalue problem (2.3) and (3.4), respectively, for $u \in H^{m+1}(\Omega)$, such that the following priori error estimates hold

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0 + \|u-u_h\|_0 \lesssim h^{k+1},\ k \geq 0 \tag{3.34}$$

$$\lambda - \lambda_h \lesssim h^{2k+2},\ k \geq 0 \tag{3.35}$$

IV. A POSTERIORI ERROR ANALYSIS

This section will present the posterior error estimator for the approximation of the feature pair $(\lambda_h, \boldsymbol{\sigma}_h, u_h)$ and prove the reliability and effectiveness of this error estimator. First of all, we will provide some explanations of the tools and notations that will be used in the proof process.

4.1. Technical tools

Let $\mathcal{E}_h = \bigcup_{\kappa \in \mathcal{T}_h} \mathcal{E}(\kappa)$, where $\mathcal{E}(\kappa)$ denotes the set of all edges in element κ, and define ω_κ and ω_e as follows:

$$\omega_\kappa = \bigcup_{\mathcal{E}(\kappa)\cap\mathcal{E}(\kappa')\neq\emptyset} \kappa' \ ,\ \omega_e = \bigcup_{e\in\mathcal{E}(\kappa')} \kappa' \tag{4.1}$$

for all edges $e \in \mathcal{E}_h$, $\mathcal{E}_h = \mathcal{E}_h^i \cup \mathcal{E}_h^b$, where $\mathcal{E}_h^i$ denotes the internal edges and $\mathcal{E}_h^b$ denotes the edges on the boundary $\partial\Omega$, let $e = \kappa_+ \cap \kappa_-$ be defined, and define the jump of the function v on edge e as:

$$J(v)|_e = (v|_{\kappa_+})\big|_e - (v|_{\kappa_-})\big|_e,\ \ e \in \mathcal{E}_h^i$$

$$J(v)|_e = (v|_\kappa)|_e,\ \ \ e \in \mathcal{E}_h^b$$

Lemma 4.1. Consider the Clément interpolation operator $\Theta_h: H^1(\Omega) \to S^1(\mathcal{T}_h)$, and define $S^1(\mathcal{T}_h) \subset H^1(\Omega)$ as the set of continuous piecewise affine functions, for any $v \in H_0^1(\Omega)$, it satisfies

$$\|v - \Theta_h v\|_{0,\kappa} \lesssim h_\kappa \|v\|_{1,\omega_\kappa} \tag{4.2}$$

$$\|v - \Theta_h v\|_{0,e} \lesssim h_e^{1/2} \|v\|_{1,\omega_e} \tag{4.3}$$

Lemma 4.2. For $\phi, \boldsymbol{\varphi} \in H^1(\Omega)$, the integration by parts formula gives

$$\int_\Omega (\boldsymbol{\varphi}\cdot \boldsymbol{curl}\phi + \phi \cdot rot\boldsymbol{\varphi})dx = \int_{\partial\Omega} \phi \cdot (\boldsymbol{\varphi}\cdot \boldsymbol{t})ds \tag{4.4}$$

where $\boldsymbol{t} = (-n_2, n_1)^T$ represents the unit outward tangent vector.

Now, we introduce the mixed finite element eigenvalue expansion [21].

Lemma 4.3. Assume $(\lambda, \boldsymbol{\sigma}, u) \in R \times \boldsymbol{H} \times V$ be the solution to the eigenvalue problem (2.3), $0 \neq \mu \in V$ and $\boldsymbol{w} \in \boldsymbol{H}$ satisfy

$$a(\boldsymbol{w}, \boldsymbol{w}) + b(\boldsymbol{w}, \mu) = 0 \tag{4.5}$$

Let us define

$$\hat{\lambda} = \frac{a(\boldsymbol{w},\boldsymbol{w})}{r(\mu,\mu)} \tag{4.6}$$

Then, we have

$$\hat{\lambda} - \lambda = \frac{-a(\boldsymbol{w}-\boldsymbol{\sigma},\boldsymbol{w}-\boldsymbol{\sigma}) - b(\boldsymbol{w}-\boldsymbol{\sigma},\mu-u)}{r(\mu,\mu)} - \frac{\lambda r(\mu-u,\mu-u)}{r(\mu,\mu)} \tag{4.7}$$

Proof. It is calculated from equations (2.8) and (4.6)

$$\begin{aligned}\hat{\lambda} - \lambda &= \frac{a(\boldsymbol{w},\boldsymbol{w}) - \lambda r(\mu,\mu)}{r(\mu,\mu)} \\ &= \frac{a(\boldsymbol{w}-\boldsymbol{\sigma},\boldsymbol{w}-\boldsymbol{\sigma}) + 2a(\boldsymbol{w},\boldsymbol{\sigma}) - a(\boldsymbol{\sigma},\boldsymbol{\sigma}) - \lambda r(\mu,\mu)}{r(\mu,\mu)} \\ &= \frac{a(\boldsymbol{w}-\boldsymbol{\sigma},\boldsymbol{w}-\boldsymbol{\sigma}) + 2a(\boldsymbol{w},\boldsymbol{\sigma}) - \lambda r(u,u) - \lambda r(\mu,\mu)}{r(\mu,\mu)} \\ &= \frac{a(\boldsymbol{w}-\boldsymbol{\sigma},\boldsymbol{w}-\boldsymbol{\sigma}) + 2a(\boldsymbol{w},\boldsymbol{\sigma}) - 2\lambda r(\mu,u) - \lambda r(\mu-u,\mu-u)}{r(\mu,\mu)} \\ &= \frac{a(\boldsymbol{w}-\boldsymbol{\sigma},\boldsymbol{w}-\boldsymbol{\sigma}) + 2a(\boldsymbol{w},\boldsymbol{\sigma}) + 2b(\boldsymbol{\sigma},\mu)}{r(\mu,\mu)} - \frac{\lambda r(\mu-u,\mu-u)}{r(\mu,\mu)}\end{aligned} \tag{4.8}$$

Using equations (2.3) and (4.5), we have

$$\begin{aligned}2a(\boldsymbol{w},\boldsymbol{\sigma})+2b(\boldsymbol{\sigma},\mu)&=2[a(\boldsymbol{w},\boldsymbol{\sigma})+b(\boldsymbol{\sigma},\mu)-a(\boldsymbol{w},\boldsymbol{w})+b(\boldsymbol{w},\mu)]\\&=-2a(\boldsymbol{w},\boldsymbol{w}-\boldsymbol{\sigma})-2b(\boldsymbol{w}-\boldsymbol{\sigma},\mu)\\&=-2a(\boldsymbol{w}-\boldsymbol{\sigma},\boldsymbol{w}-\boldsymbol{\sigma})-2b(\boldsymbol{w}-\boldsymbol{\sigma},\mu-u)\end{aligned}\tag{4.9}$$

Finally, by combining equations (4.8) and (4.9), equation (4.7) can be obtained.

Next, by Lemma 4.3, and in conjunction with equations (2.8) and (3.5), the following relationship between the eigenvalue error and the eigenfunction error can be obtained.

Lemma 4.4. Let $(\lambda,\boldsymbol{\sigma},u)\in R\times\boldsymbol{H}\times V$ be the solution to problem (2.3), and $(\lambda_h,\boldsymbol{\sigma}_h,u_h)\in R\times\boldsymbol{H}_h\times V_h$ be the solution to the finite element approximation (3.4), then the following equation holds

$$|\lambda-\lambda_h|^{1/2}\lesssim\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\boldsymbol{H}}+\|u-u_h\|_0\tag{4.10}$$

4.2. Local and global error indicators

Define the local error indicator on each element κ.

$$\eta_\kappa^2=h_\kappa^2\|rot(c^{-1}\boldsymbol{\sigma}_h)\|_{0,\kappa}^2+h_\kappa^2\|c^{-1}\boldsymbol{\sigma}_h-\nabla v_h\|_{0,\kappa}^2+h_e\|J(c^{-1}\boldsymbol{\sigma}_h\cdot\boldsymbol{t})\|_{0,e}^2\tag{4.11}$$

The global error indicator is given by

$$\eta_h=\left(\sum_{\kappa\in\mathcal{T}_h}\eta_\kappa^2\right)^{1/2}\tag{4.12}$$

Next, we will prove that this error estimate is reliable.

4.3. The reliability of the eigenfunction estimator

Theorem 4.1. Let $(\lambda,\boldsymbol{\sigma},u)\in R\times\boldsymbol{H}\times V$ be the solution of problem (2.3), and $(\lambda_h,\boldsymbol{\sigma}_h,u_h)\in R\times\boldsymbol{H}_h\times V_h$ be the solution of the finite element approximation (3.4), then we have

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\boldsymbol{H}}+\|u-u_h\|_V\lesssim\eta_h+h.o.t\tag{4.13}$$

where $h.o.t=\left(\sum_{\kappa\in\mathcal{T}_h}h_\kappa^2(|\lambda-\lambda_h|^2+\|u-u_h\|_0^2)\right)^{1/2}$

For Theorem 4.1 above, we will prove it in the following two lemmas.

Lemma 4.5. The following estimate holds

$$\left\|c^{-1/2}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\right\|_{0,\Omega}\lesssim\left(h^2\|rot(c^{-1}\boldsymbol{\sigma}_h)\|_{0,\Omega}^2+h\left\|J\left(c^{-1/2}\boldsymbol{\sigma}_h\cdot\boldsymbol{t}\right)\right\|_{0,\mathcal{E}_h}^2\right)^{1/2}+h.o.t\tag{4.14}$$

where the definition of $h.o.t$ can be found in Theorem 4.1.

Proof. Let $\rho\in H_0^1(\Omega)$, and consider the orthogonal Helmholtz decomposition of $c^{-1}\boldsymbol{\sigma}_h$

$$div(c\cdot\nabla\rho)=div\boldsymbol{\sigma}_h\tag{4.15}$$

Then, there exists $\beta\in H^1(\Omega)$ such that $\int_\Omega\beta dx=0$, $\boldsymbol{curl}\beta\perp\nabla H_0^1(\Omega)$, and

$$\boldsymbol{\sigma}_h=c\cdot\nabla\rho+\boldsymbol{curl}\beta\tag{4.16}$$

From equation (2.2) and equation (4.16), we can obtain

$$\boldsymbol{\sigma}-\boldsymbol{\sigma}_h=c\cdot\nabla\mathcal{X}-\boldsymbol{curl}\beta,\ \mathcal{X}=u-\rho\in H_0^1(\Omega)\tag{4.17}$$

Therefore, there is an error decomposition

$$\begin{aligned}\int_\Omega c^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\cdot(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)dx&=\int_\Omega(\nabla\mathcal{X}-c^{-1}\boldsymbol{curl}\beta)\cdot(c\cdot\nabla\mathcal{X}-\boldsymbol{curl}\beta)dx\\&=\int_\Omega(c\cdot\nabla\mathcal{X})\cdot\nabla\mathcal{X}dx+\int_\Omega(c^{-1}\boldsymbol{curl}\beta)\cdot\boldsymbol{curl}\beta dx\end{aligned}\tag{4.18}$$

Firstly, we estimate the first term $\int_\Omega(c\cdot\nabla\mathcal{X})\cdot\nabla\mathcal{X}dx$. Using integration by parts, equation (2.2), and $div(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\perp V_h$, we can obtain that

$$\begin{aligned}\int_\Omega(c\cdot\nabla\mathcal{X})\nabla\mathcal{X}dx&=\int_\Omega\nabla\mathcal{X}\cdot(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)dx+\int_\Omega\nabla\mathcal{X}\cdot\boldsymbol{curl}\beta dx\\&=\int_\Omega\nabla\mathcal{X}\cdot(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)dx\\&=-\int_\Omega(\mathcal{X}-\Pi_h\mathcal{X})\cdot div(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)dx\\&\lesssim h\|div(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_0\left\|c^{1/2}\cdot\nabla\mathcal{X}\right\|_0\\&\lesssim h\|\lambda_hu_h-\lambda u\|_0\left\|c^{1/2}\cdot\nabla\mathcal{X}\right\|_0\end{aligned}\tag{4.19}$$

Next, we estimate the second term $\int_\Omega(c^{-1}\boldsymbol{curl}\beta)\cdot\boldsymbol{curl}\beta dx$. We use the Clément operator Θ_h, defined as $\beta_h=\Theta_h\beta$, which satisfies the properties $\boldsymbol{curl}\beta_h\perp\nabla H_0^1(\Omega)$ and $div\boldsymbol{curl}\beta_h=0$. Then, based on equations (3.3), (4.17), (2.2), and (3.4), we obtain that

$$\begin{aligned}\int_\Omega (c^{-1}\boldsymbol{curl}\beta)\cdot \boldsymbol{curl}\beta_h dx &= -\int_\Omega c^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\cdot \boldsymbol{curl}\beta_h dx + \int_\Omega \nabla\mathcal{X}\cdot \boldsymbol{curl}\beta_h dx\\ &= -\int_\Omega \nabla(u-u_h)\cdot \boldsymbol{curl}\beta_h dx\\ &= \int_\Omega (u-u_h)\cdot div\boldsymbol{curl}\beta_h dx\\ &= 0\end{aligned} \tag{4.20}$$

By Lemma 4.2, using equations (4.20) and (4.16), it can be derived that

$$\begin{aligned}\int_\Omega (c^{-1}\boldsymbol{curl}\beta)\cdot \boldsymbol{curl}\beta dx &= \int_\Omega (c^{-1}\boldsymbol{curl}\beta)\cdot \boldsymbol{curl}(\beta-\beta_h)dx\\ &= \int_\Omega c^{-1}\boldsymbol{\sigma}_h\cdot \boldsymbol{curl}(\beta-\beta_h)dx - \int_\Omega \nabla\rho\cdot \boldsymbol{curl}(\beta-\beta_h)dx\\ &= \int_\Omega c^{-1}\boldsymbol{\sigma}_h\cdot \boldsymbol{curl}(\beta-\beta_h)dx\\ &= -\int_\Omega (\beta-\beta_h)\cdot rot(c^{-1}\boldsymbol{\sigma}_h)dx + \int_{\mathcal{E}_h} J(c^{-1}\boldsymbol{\sigma}_h\cdot \boldsymbol{t})\cdot(\beta-\beta_h)ds\end{aligned} \tag{4.21}$$

Based on equations (4.2) and (4.3), along with $\beta_h = \Theta_h\beta$, the following estimate can be obtained

$$\int_\Omega (\beta-\beta_h)\cdot rot(c^{-1}\boldsymbol{\sigma}_h)dx \lesssim \|h\cdot rot(c^{-1}\boldsymbol{\sigma}_h)\|_0\|\beta\|_1 \tag{4.22}$$

$$\int_{\mathcal{E}_h} J(c^{-1}\boldsymbol{\sigma}_h\cdot \boldsymbol{t})\cdot(\beta-\beta_h)ds \lesssim \left\|h^{1/2}\cdot J(c^{-1}\boldsymbol{\sigma}_h\cdot \boldsymbol{t})\right\|_0\|\beta\|_1 \tag{4.23}$$

By Poincaré's inequality (equation (1.2)), we have

$$\|\beta\|_1 \lesssim \|\nabla\beta\|_0 = \|\boldsymbol{curl}\beta\|_0 \lesssim \left\|c^{-1/2}\boldsymbol{curl}\beta\right\|_0 \tag{4.24}$$

Therefore, the following estimate can be obtained

$$\int_\Omega (c^{-1}\boldsymbol{curl}\beta)\cdot \boldsymbol{curl}\beta dx \lesssim \left\|c^{-1/2}\boldsymbol{curl}\beta\right\|_0\left(\left\|h^{1/2}\cdot J(c^{-1}\boldsymbol{\sigma}_h\cdot \boldsymbol{t})\right\|_0 + \|h\cdot rot(c^{-1}\boldsymbol{\sigma}_h)\|_0\right) \tag{4.25}$$

By equations (4.18), (4.19), and (4.25), along with $div(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h) = \lambda_h u_h - \lambda u$, the following estimate can be derived

$$\left\|c^{-1/2}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\right\|_0 \lesssim \left(h^2\|\lambda_h u_h - \lambda u\|_{0,\Omega}^2 + h^2\|rot(c^{-1}\boldsymbol{\sigma}_h)\|_{0,\Omega}^2 + h\left\|J\left(c^{-1/2}\boldsymbol{\sigma}_h\cdot \boldsymbol{t}\right)\right\|_{0,\mathcal{E}_h}^2\right)^{1/2} \tag{4.26}$$

Finally, Lemma 4.5 is proven.

Lemma 4.6. The following estimate holds

$$\|u-u_h\|_0 \lesssim (h^2\|c^{-1}\boldsymbol{\sigma}_h - \nabla v_h\|_0^2)^{1/2} + h.o.t \tag{4.27}$$

where the definition of $h.o.t$ can be found in Theorem 4.1.

Proof. There exists $\theta \in H_0^1(\Omega)$ such that $div(c\cdot\nabla\theta) = u-u_h$. By integration by parts, equations (2.2), (3.4), and (3.9), for any $v_h \in V_h$, we have

$$\begin{aligned}\|u-u_h\|_0^2 &= \int_\Omega (u-u_h)\cdot div(c\cdot\nabla\theta)dx\\ &= \int_\Omega u\cdot div(c\cdot\nabla\theta)dx - \int_\Omega u_h\cdot div(c\cdot\nabla\theta)dx\\ &= -\int_\Omega \nabla u\cdot(c\cdot\nabla\theta)dx - \int_\Omega u_h\cdot div\boldsymbol{Q}_h(c\cdot\nabla\theta)dx\\ &\quad - \int_\Omega u_h\cdot div(Id-\boldsymbol{Q}_h)(c\cdot\nabla\theta)dx\\ &= -\int_\Omega \boldsymbol{\sigma}\cdot\nabla\theta dx + \int_\Omega (c^{-1}\boldsymbol{\sigma}_h)\cdot \boldsymbol{Q}_h(c\cdot\nabla\theta)dx\\ &= -\int_\Omega (\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\cdot\nabla\theta dx - \int_\Omega (c^{-1}\boldsymbol{\sigma}_h)\cdot(Id-\boldsymbol{Q}_h)(c\cdot\nabla\theta)dx\\ &= \int_\Omega \theta\cdot div(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)dx + \int_\Omega (\nabla v_h - c^{-1}\boldsymbol{\sigma}_h)\cdot(Id-\boldsymbol{Q}_h)(c\cdot\nabla\theta)dx\\ &\quad - \int_\Omega \nabla v_h\cdot(Id-\boldsymbol{Q}_h)(c\cdot\nabla\theta)dx\end{aligned} \tag{4.28}$$

Let $\theta_h = \Pi_h\theta$, since $\|\theta_h\|_0 = \|\Pi_h\theta\|_0 \lesssim \|\theta\|_0$ is an infinitesimal, it follows from equations (2.2), (3.4), and (3.6) that

$$\begin{aligned}\int_\Omega \theta\cdot div(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)dx &= \int_\Omega (\theta-\Pi_h\theta)\cdot div(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)dx + \int_\Omega \theta_h\cdot div(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)dx\\ &\lesssim \|\nabla\theta\|_0\|div(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_0 + \|\theta\|_0\|div(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_0\\ &\lesssim \|\nabla\theta\|_0\|div(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_0\end{aligned} \tag{4.29}$$

Furthermore, the second term on the right-hand side of equation (4.28) can be estimated as

$$\int_\Omega (\nabla v_h - c^{-1}\boldsymbol{\sigma}_h)\cdot(Id-\boldsymbol{Q}_h)(c\cdot\nabla\theta)dx \lesssim \|h(\nabla v_h - c^{-1}\boldsymbol{\sigma}_h)\|_0\|h^{-1}(Id-\boldsymbol{Q}_h)(c\cdot\nabla\theta)\|_0 \tag{4.30}$$

According to equation (3.10), we have

$$\|h^{-1}(Id-\boldsymbol{Q}_h)(c\cdot\nabla\theta)\|_0 \lesssim |c\cdot\nabla\theta|_1 \lesssim \|\theta\|_2 \tag{4.31}$$

Finally, using equations (3.9) and (3.11), the last term on the right-hand side of equation (4.28) can be estimated as

$$\begin{aligned}\int_{\Omega}\nabla v_h\cdot(Id-\boldsymbol{Q}_h)(c\cdot\nabla\theta)dx &= -\int_{\Omega}v_h\cdot div(Id-\boldsymbol{Q}_h)(c\cdot\nabla\theta)dx\\ &\quad+\int_{\partial\Omega}v_h\cdot(Id-\boldsymbol{Q}_h)(c\cdot\nabla\theta)\cdot\boldsymbol{n}ds\\ &=0\end{aligned}\tag{4.32}$$

In conclusion, by combining equations (4.28) to (4.32), we obtain

$$\begin{aligned}\|u-u_h\|_0^2 &\lesssim \|u-u_h\|_0(h^2\|(\nabla v_h-c^{-1}\boldsymbol{\sigma}_h)\|_0^2+h^2\|div(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_0^2)^{1/2}\\ &\lesssim \|u-u_h\|_0(h^2\|(\nabla v_h-c^{-1}\boldsymbol{\sigma}_h)\|_0^2+h^2\|\lambda_h u_h-\lambda u\|_0^2)^{1/2}\end{aligned}\tag{4.33}$$

By simplifying, we can prove Lemma 4.6.

From the prior estimates given by equations (3.34) and (3.35), we know that $\|\lambda_h u_h-\lambda u\|_0$ is a higher-order small term relative to $\|u-u_h\|_0$. Therefore, equation (4.13) tells us that the error estimate indicator is one of the upper bounds of the error estimate, and thus, it is reliable.

4.4. The effectiveness of the eigenfunction estimator

The following task is to prove the lower bound of the local error estimator. This lower bound is obtained based on the localization technique of bubble functions and inverse inequalities.

We first introduce the bubble functions in two-dimensional space. Given $\kappa\in\mathcal{T}_h$ and $e\in\mathcal{E}_h$, let ψ_κ be the standard element bubble function, and ψ_e be the bubble function on the surface. By utilizing the bubble function technique developed by Verfürth [22], the following properties are satisfied:

1. $\psi_\kappa\in\mathcal{P}_3(\kappa)$, with $0\le\psi_\kappa\le 1=max\,\psi_\kappa$ in κ, $\psi_\kappa=0$ on $\partial\kappa$;
2. $\psi_e\in\omega_e$, with $0\le\psi_e\le 1=max\,\psi_e$, $\forall\kappa\in\mathcal{T}_h$, $\kappa\subset\omega_E$, it satisfies $\psi_e|_\kappa\in\mathcal{P}_2(\kappa)$.

Lemma 4.7. For any $\kappa\in\mathcal{T}_h$ and $e\in\mathcal{E}_h$, the following equation holds

$$\|v\|_{0,\kappa}\lesssim\left\|\psi_\kappa^{1/2}v\right\|_{0,\kappa}\quad\forall v\in\mathcal{P}_k(\kappa)\tag{4.34}$$

$$\|\xi\|_{0,e}\lesssim\left\|\psi_e^{1/2}\xi\right\|_{0,e}\quad\forall\xi\in\mathcal{P}_k(e)\tag{4.35}$$

For each $\psi_e\xi$, there exists an extension factor $\mathcal{B}:\mathcal{C}(e)\to\mathcal{C}(\omega_e)$, where $\mathcal{C}(e)$ and $\mathcal{C}(\omega_e)$ are continuous function spaces defined on e and ω_e, respectively, such that $\mathcal{B}\xi|_e=\xi$, and

$$h_e^{1/2}\|\xi\|_{0,e}\lesssim\left\|\psi_e^{1/2}\cdot\mathcal{B}\xi\right\|_{0,\omega_e}\lesssim h_e^{1/2}\|\xi\|_{0,e}\tag{4.36}$$

Lemma 4.8. Let $(\lambda,\boldsymbol{\sigma},u)\in R\times\boldsymbol{H}\times V$ be the solution of (2.3), and $(\lambda_{\square},\boldsymbol{\sigma}_{\square},u_{\square})\in R\times\boldsymbol{H}_{\square}\times V_{\square}$ be the solution of the finite element approximation (3.4), then we have the local lower bound as follows

(i) For any $\kappa\in\mathcal{T}_h$ and β, there holds

$$h_\kappa\|rot(c^{-1}\boldsymbol{\sigma}_h)\|_{0,\kappa}\lesssim\left\|c^{-1/2}\boldsymbol{curl}\beta\right\|_{0,\kappa}\tag{4.37}$$

(ii) For any $\kappa\in\mathcal{T}_h$, there holds

$$h_\kappa\|c^{-1}\boldsymbol{\sigma}_h-\nabla u_h\|_{0,\kappa}\lesssim\|u-u_h\|_{0,\kappa}+h_\kappa\left\|c^{-1/2}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\right\|_{0,\kappa}\tag{4.38}$$

(iii) For any $e\in\mathcal{E}_h$, there holds

$$h_e^{1/2}\|J(c^{-1}\boldsymbol{\sigma}_h\cdot\boldsymbol{t})\|_{0,e}\lesssim\left\|c^{-1/2}\boldsymbol{curl}\beta\right\|_{0,\omega_e}\tag{4.39}$$

Proof. (i) Firstly, from equation (4.34), the following equation holds

$$\|rot(c^{-1}\boldsymbol{\sigma}_h)\|_{0,\kappa}^2\lesssim\left\|\psi_\kappa^{1/2}\cdot rot(c^{-1}\boldsymbol{\sigma}_h)\right\|_{0,\kappa}^2\tag{4.40}$$

By integration by parts, we have $rot(c^{-1}\boldsymbol{\sigma}_h)=-rot\big(c^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\big)$, and combining this with equation (4.17), we can deduce that

$$\begin{aligned}\left\|\psi_\kappa^{1/2}\cdot rot(c^{-1}\boldsymbol{\sigma}_h)\right\|_{0,\kappa}^2 &= -\int_\kappa\psi_\kappa\cdot rot(c^{-1}\boldsymbol{\sigma}_h)\cdot rot\big(c^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\big)dx\\ &=\int_\kappa\big(c^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\big)\cdot\boldsymbol{curl}\big(\psi_\kappa\cdot rot(c^{-1}\boldsymbol{\sigma}_h)\big)dx\\ &=\int_\kappa(\nabla\mathcal{X}-c^{-1}\boldsymbol{curl}\beta)\cdot\boldsymbol{curl}\big(\psi_\kappa\cdot rot(c^{-1}\boldsymbol{\sigma}_h)\big)dx\\ &=-\int_\kappa(c^{-1}\boldsymbol{curl}\beta)\cdot\boldsymbol{curl}\big(\psi_\kappa\cdot rot(c^{-1}\boldsymbol{\sigma}_h)\big)dx\end{aligned}\tag{4.41}$$

Since $\psi_\kappa\cdot rot(c^{-1}\boldsymbol{\sigma}_h)\in\mathcal{P}_{l+2}$ has zero boundary values on κ, then we have

$$|\psi_\kappa\cdot rot(c^{-1}\boldsymbol{\sigma}_h)|_{1,\kappa}\lesssim h_\kappa^{-1}\|\psi_\kappa\cdot rot(c^{-1}\boldsymbol{\sigma}_h)\|_{0,\kappa}\tag{4.42}$$

Finally, by applying the Cauchy-Schwarz inequality and using equations (4.40) to (4.42), we can derive that

$$\left\|\psi_\kappa^{1/2}\cdot rot(c^{-1}\boldsymbol{\sigma}_h)\right\|_{0,\kappa}\lesssim h_\kappa^{-1}\left\|c^{-1/2}\boldsymbol{curl}\beta\right\|_{0,\kappa}\tag{4.43}$$

Thus, (i) is proven.

(ii) For any $\kappa \in \mathcal{T}_h$, from equation (4.34), we have

$$\|c^{-1}\boldsymbol{\sigma}_h - \nabla u_h\|_{0,\kappa}^2 \lesssim \left\|\psi_\kappa^{1/2} \cdot (c^{-1}\boldsymbol{\sigma}_h - \nabla u_h)\right\|_{0,\kappa}^2 \tag{4.44}$$

By using the integration by parts formula and $c^{-1}\boldsymbol{\sigma} = \nabla u$, we can obtain

$$\begin{aligned}\left\|\psi_\kappa^{1/2} \cdot (c^{-1}\boldsymbol{\sigma}_h - \nabla u_h)\right\|_{0,\kappa}^2 &= \int_\kappa \psi_\kappa \cdot (c^{-1}\boldsymbol{\sigma}_h - \nabla u_h) \cdot (c^{-1}\boldsymbol{\sigma}_h - \nabla u_h)dx \\ &= \int_\kappa \psi_\kappa \cdot c^{-1}\boldsymbol{\sigma}_h \cdot (c^{-1}\boldsymbol{\sigma}_h - \nabla u_h)dx \\ &\quad - \int_\kappa \psi_\kappa \cdot \nabla u_h \cdot (c^{-1}\boldsymbol{\sigma}_h - \nabla u_h)dx \\ &= -\int_\kappa \psi_\kappa \cdot c^{-1}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) \cdot (c^{-1}\boldsymbol{\sigma}_h - \nabla u_h)dx \\ &\quad - \int_\kappa (u - u_h) \cdot div\big(\psi_\kappa(c^{-1}\boldsymbol{\sigma}_h - \nabla u_h)\big)dx \\ &\lesssim \|c^{-1}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,\kappa}\|\psi_\kappa(c^{-1}\boldsymbol{\sigma}_h - \nabla u_h)\|_{0,\kappa} \\ &\quad + \|u - u_h\|_{0,\kappa}|\psi_\kappa(c^{-1}\boldsymbol{\sigma}_h - \nabla u_h)|_{1,\kappa}\end{aligned} \tag{4.45}$$

Similar to equation (4.43), we can derive

$$|\psi_\kappa(c^{-1}\boldsymbol{\sigma}_h - \nabla u_h)|_{1,\kappa} \lesssim h_\kappa^{-1}\|\psi_\kappa(c^{-1}\boldsymbol{\sigma}_h - \nabla u_h)\|_{0,\kappa} \tag{4.46}$$

Finally, from equations (4.45) and (4.46), we have

$$\left\|\psi_\kappa^{1/2} \cdot (c^{-1}\boldsymbol{\sigma}_h - \nabla u_h)\right\|_{0,\kappa} \lesssim \left\|c^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\right\|_{0,\kappa} + h_\kappa^{-1}\|u - u_h\|_{0,\kappa} \tag{4.47}$$

By combining equations (4.44) and (4.47), we can prove (ii).

(iii) For any $e \in \mathcal{E}_h$, let $\delta = J(c^{-1}\boldsymbol{\sigma}_h \cdot \boldsymbol{t})$ be a polynomial of degree $\leq k$ along the edge e. By using the properties of the bubble function ψ_e, and combining with equation (4.35), we have

$$\|\delta\|_{0,e}^2 \lesssim \left\|\psi_e^{1/2}\delta\right\|_{0,e}^2 = \int_e \psi_e \delta \cdot J(c^{-1}\boldsymbol{\sigma}_h \cdot \boldsymbol{t})ds = -\int_e \psi_e \cdot \mathcal{B}\delta \cdot J(c^{-1}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) \cdot \boldsymbol{t})ds \tag{4.48}$$

By applying equation (4.4) to each element κ, and combining with equation (4.17), we can obtain

$$\begin{aligned}-\int_e \psi_e \cdot \mathcal{B}\delta \cdot J(c^{-1}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) \cdot \boldsymbol{t})ds &= -\int_{\omega_e} c^{-1}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) \cdot \boldsymbol{curl}(\psi_e \cdot \mathcal{B}\delta)dx \\ &\quad - \int_{\omega_e} (\psi_e \cdot \mathcal{B}\delta) \cdot rot\big(c^{-1}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\big)dx \\ &= \int_{\omega_e} c^{-1}\boldsymbol{curl}\beta \cdot \boldsymbol{curl}(\psi_e \cdot \mathcal{B}\delta)dx \\ &\quad + \int_{\omega_e} (\psi_e \cdot \mathcal{B}\delta) \cdot rot(c^{-1}\boldsymbol{\sigma}_h)dx \\ &\lesssim \|c^{-1}\boldsymbol{curl}\beta\|_{0,\omega_e}|\psi_e \cdot \mathcal{B}\delta|_{1,\omega_e} \\ &\quad + \|\psi_e \cdot \mathcal{B}\delta\|_{0,\omega_e}\|rot(c^{-1}\boldsymbol{\sigma}_h)\|_{0,\omega_e}\end{aligned} \tag{4.49}$$

From (i) and equation (4.36), we can deduce that

$$\begin{aligned}-\int_e \psi_e \cdot \mathcal{B}\delta \cdot J(c^{-1}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) \cdot \boldsymbol{t})ds &\lesssim \|c^{-1}\boldsymbol{curl}\beta\|_{0,\omega_e}|\psi_e \cdot \mathcal{B}\delta|_{1,\omega_e} \\ &\quad + h_e^{-1/2}\|\delta\|_{0,e}\left\|c^{-1/2}\boldsymbol{curl}\beta\right\|_{0,\omega_e}\end{aligned} \tag{4.50}$$

Since $\psi_e \cdot \mathcal{B}\delta$ is an extension of polynomials, there is an inverse inequality (for details, refer to [23])

$$|\psi_e \cdot \mathcal{B}\delta|_{1,\omega_e} \lesssim h_e^{-1}\|\psi_e \cdot \mathcal{B}\delta\|_{0,\omega_e} \tag{4.51}$$

By further using equation (4.36), we can obtain

$$|\psi_e \cdot \mathcal{B}\delta|_{1,\omega_e} \lesssim h_e^{-1/2}\|\delta\|_{0,e} \tag{4.52}$$

Finally, by using equations (4.48), (4.50), and (4.52), we can deduce that

$$\|\delta\|_{0,e} \lesssim h_e^{-1/2}\|c^{-1}\boldsymbol{curl}\beta\|_{0,\omega_e} \tag{4.53}$$

Thus, we can obtain (iii).

In conclusion, Lemma 4.8 is proved.

Theorem 4.2. Under the conditions of Lemma 4.8, the following estimate holds

$$\eta_h = \left(\sum_{\kappa \in \mathcal{T}_h} \eta_\kappa^2\right)^{1/2} \lesssim \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\boldsymbol{H}} + \|u - u_h\|_V + h.o.t \tag{4.54}$$

where the definition of $h.o.t$ can be found in Theorem 4.1.

Proof. The proof follows directly from the definition of η_κ and Lemma 4.8.

V. NUMERICAL EXAMPLE

In this section, some numerical experiments will be presented to demonstrate the effectiveness of the method. Here we give the numerical results of the adaptive mixed finite element algorithm for the first eigenpair approximation with the parameter $\theta = 0.4$. For problem (2.1), we consider three cases with

$$c(x)=1,\ c(x)=\frac{1}{1+(x-1/2)^2},\ \text{and}\ c(x)=\frac{1}{1+x^2y^2}.$$

The corresponding numerical results are shown in the tables and figures. Additionally, the numerical examples in this paper were computed using MATLAB 2020b under the iFEM software package (see [18]).

In the experiment, we consider two test domains: the L-shaped domain $\Omega_L=(-1,1)^2\backslash(0,1)\times -1,0)$,and the crack structure domain $\Omega_{SL}=(-1,1)^2\backslash\{0\le x\le 1, y=0\}$. Since the exact eigenvalues are unknown, we select six sufficiently accurate approximate values as the reference for the numerical test. These reference eigenvalues are obtained as accurately as possible through adaptive computations. The specific results are as follows:

TABLE.1. When $c(x)=1$, the numerical solution for the eigenvalues on the uniform grid regions Ω_L,Ω_{SL}.

$domain$	ref	l	h	dof	λ_1	$Error$	$rate$
Ω_L	9.6397238440	13	1/4	3709	9.5807580032	0.0756736084	1.6008780762
		13	1/8	11699	9.6182024994	0.0249477925	1.8930641442
		13	1/16	47494	9.6342625921	0.0067168095	1.9587441726
		13	1/32	186429	9.6383032538	0.0017279146	2.0035999298
		13	1/64	752661	9.6393491891	0.0004309021	1.9837704971
		13	1/128	2995689	9.6396282237	0.0001089442	
$domain$	ref	l	h	dof	λ_1	$Error$	$rate$
Ω_{SL}	8.3713297112	12	1/4	3819	8.2679577933	0.1010843246	1.6621284915
		12	1/8	12978	8.3352881476	0.0319398801	1.7142243455
		12	1/16	48012	8.3601806892	0.0097342015	1.9483208464
		12	1/32	185924	8.3675732991	0.0025223034	1.9896648862
		12	1/64	740944	8.3699203051	0.0006351094	1.9948796783
		12	1/128	2966924	8.3707447047	0.0001593419	

TABLE.2. When $c(x)=\frac{1}{1+(x-1/2)^2}$, the numerical solution for the eigenvalues on the uniform grid regions Ω_L,Ω_{SL}.

$domain$	ref	l	h	dof	λ_1	$Error$	$rate$
Ω_L	5.3470894509	13	1/4	4052	5.3061702130	0.0572259904	1.8241265963
		13	1/8	14443	5.3389337936	0.0161613076	1.8097851098
		13	1/16	49396	5.3445689769	0.0046097449	1.9465451641
		13	1/32	192894	5.3463499851	0.0011959373	1.9577700951
		13	1/64	769105	5.3469668467	0.0003078654	1.9782497080
		13	1/128	3040068	5.3471123190	0.0000781355	
$domain$	ref	l	h	dof	λ_1	$Error$	$rate$
Ω_{SL}	4.7612704287	13	1/4	3784	4.6904214480	0.0716820469	2.1060054485
		13	1/8	16118	4.7450729963	0.0166509719	1.7228447757
		13	1/16	56517	4.7562515902	0.0050444246	1.9750537357
		13	1/32	224885	4.7597952414	0.0012831021	1.9609715371
		13	1/64	898331	4.7609003213	0.0003295717	1.9581192698
		13	1/128	3561025	4.7612542907	0.0000848198	

TABLE.3. When $c(x)=\frac{1}{1+x^2y^2}$, the numerical solution for the eigenvalues on the uniform grid regions Ω_L,Ω_{SL}.

$domain$	ref	l	h	dof	λ_1	$Error$	$rate$
Ω_L	9.0569153630	14	1/4	5666	9.0112319202	0.0618039421	1.5688891144
		14	1/8	17675	9.0426574110	0.0208321212	1.8799588051
		14	1/16	65154	9.0525628930	0.0056599097	1.9536618226
		14	1/32	256232	9.0558243736	0.0014611631	1.9795483080
		14	1/64	1032222	9.0567166714	0.0003705060	1.9758396980
		14	1/128	4116138	9.0569460689	0.0000941908	
$domain$	ref	l	h	dof	λ_1	$Error$	$rate$
Ω_{SL}	7.8686199409	13	1/4	4812	7.7848435143	0.0926331259	1.7346819229
		13	1/8	16791	7.8401260386	0.0278339619	1.8316428408
		13	1/16	64226	7.8597321697	0.0078197990	1.8611591162
		13	1/32	245248	7.8659188973	0.0021524391	1.9351071299
		13	1/64	979113	7.8678291729	0.0005628667	1.9763531233
		13	1/128	3907617	7.8685169952	0.0001430421	

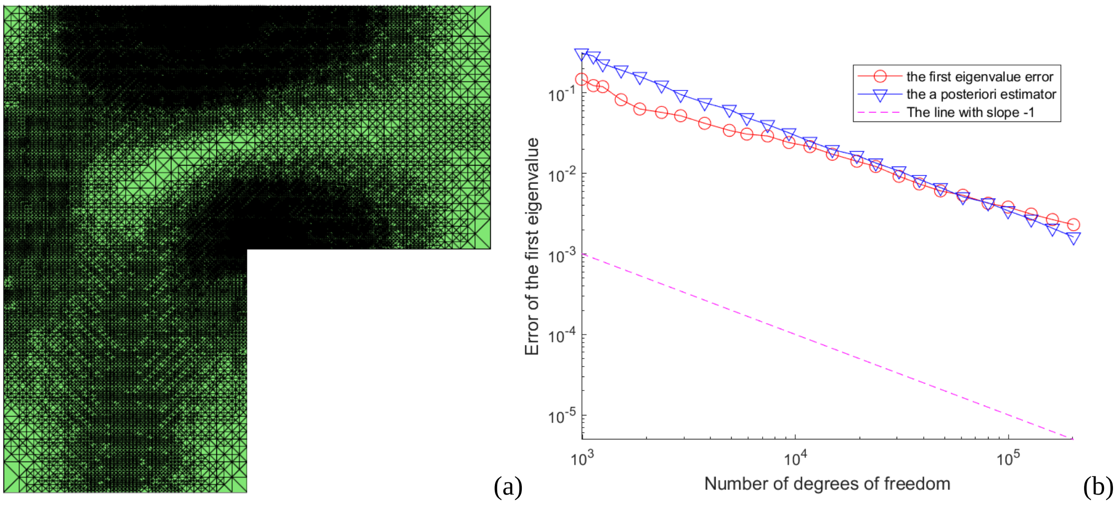

Figure.1. When $c(x) = 1$, the adaptive mesh and error curve plot on the initial grid 1/8 for the test domain Ω_L.
(a) Mesh after 25 iterations; (b) The error curve plot

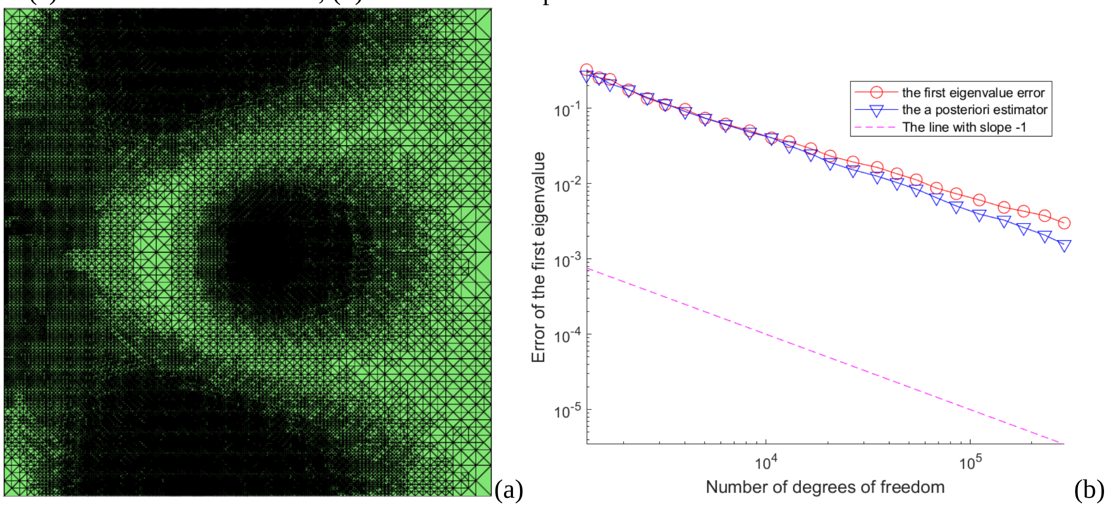

Figure.2. When $c(x) = 1$, the adaptive mesh and error curve plot on the initial grid 1/8 for the test domain Ω_{SL}.
(a) Mesh after 25 iterations; (b) The error curve plot

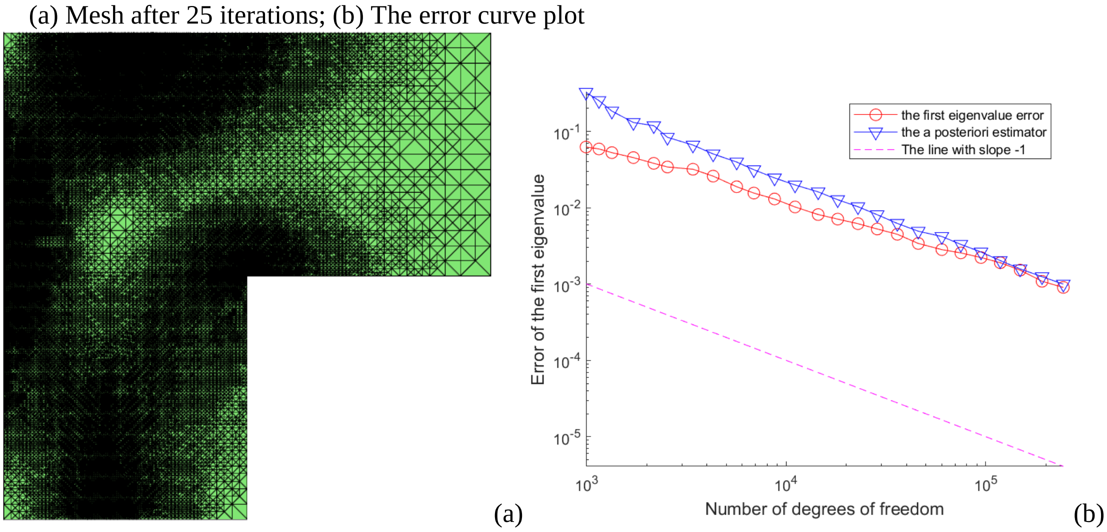

Figure.3. When $c(x) = \frac{1}{1+(x-1/2)^2}$, the adaptive mesh and error curve plot on the initial grid 1/8 for the test domain Ω_L
(a) Mesh after 25 iterations; (b) The error curve plot

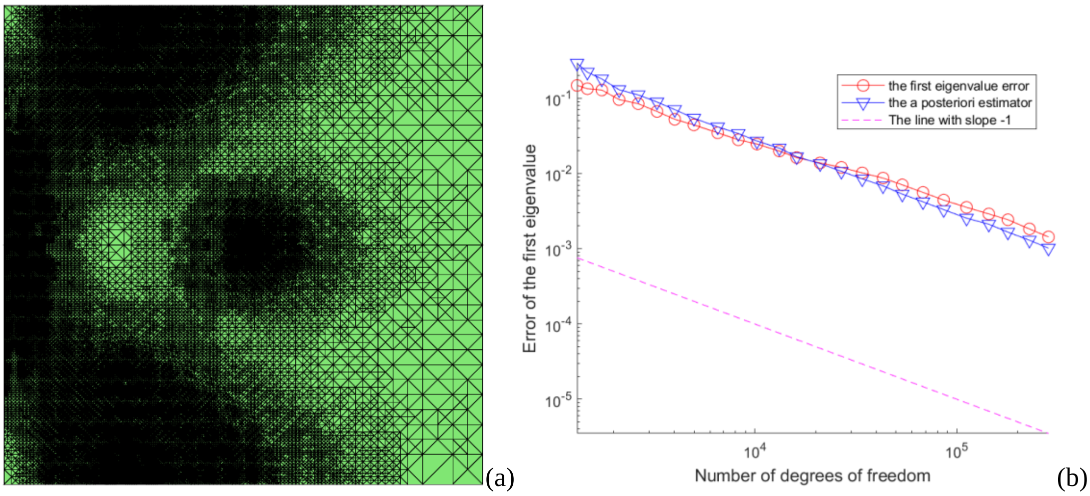

Figure.4. When $c(x) = \frac{1}{1+(x-1/2)^2}$, the adaptive mesh and error curve plot on the initial grid 1/8 for the test domain Ω_{SL}

(a) Mesh after 25 iterations; (b) The error curve plot

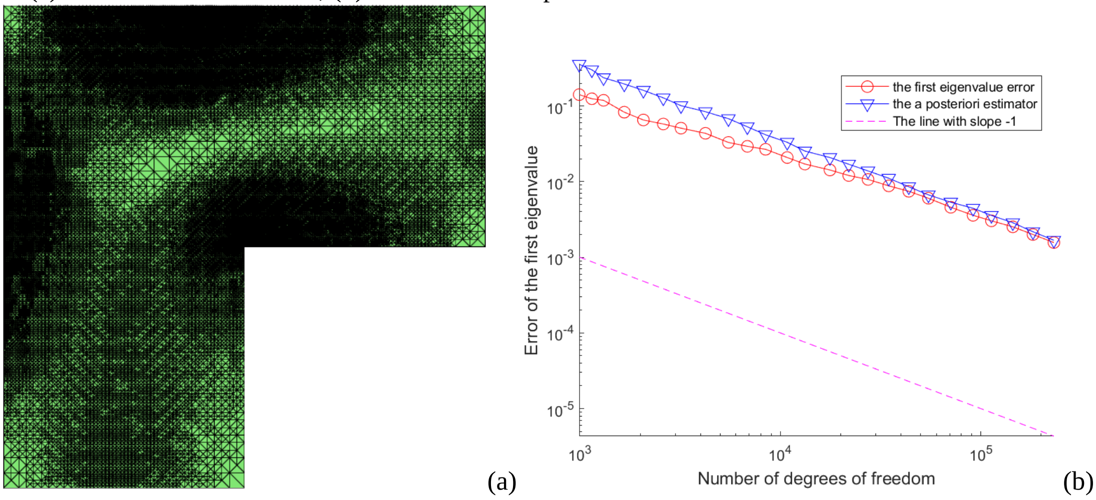

Figure.5. When $c(x) = \frac{1}{1+x^2y^2}$, the adaptive mesh and error curve plot on the initial grid 1/8 for the test domain Ω_L

(a) Mesh after 25 iterations; (b) The error curve plot

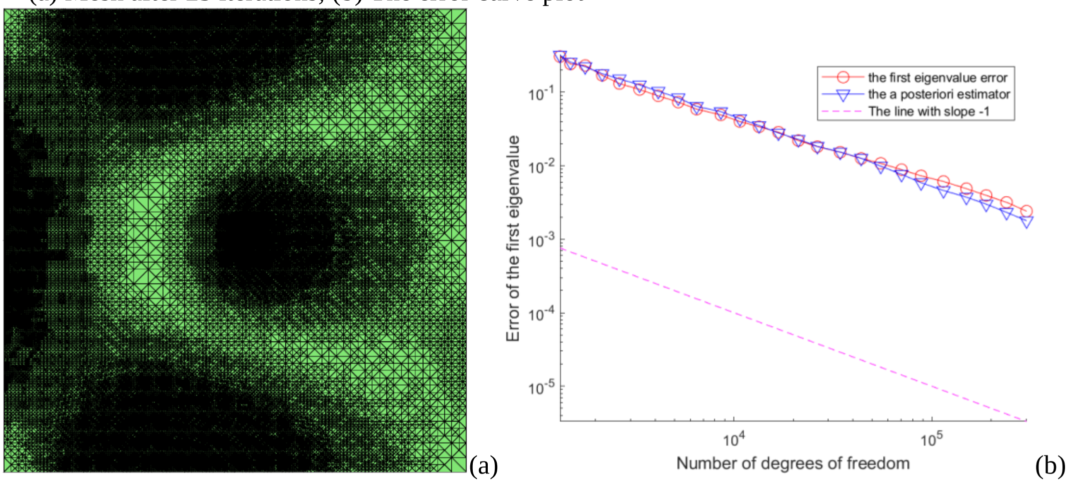

Figure.6. When $c(x) = \frac{1}{1+x^2y^2}$, the adaptive mesh and error curve plot on the initial grid 1/8 for the test domain Ω_{SL}

(a) Mesh after 25 iterations; (b) The error curve plot

VI. CONCLUSION

We present the numerical solutions of the eigenvalues obtained through the adaptive algorithm in Tables.1 to 3, and show the adaptive meshes and error curves in the figures. From Fig.1 to 6, it can be observed that when $c(x) = 1$, $c(x) = \frac{1}{1+(x-1/2)^2}$, and $c(x) = \frac{1}{1+x^2y^2}$, the error curves are approximately parallel to a straight line with a slope of -1. The results indicate that the adaptive algorithm achieves optimal convergence order. Furthermore, the error curves also show that, for the same degrees of freedom, the approximations obtained by the adaptive algorithm are significantly more accurate than those obtained using a uniform grid.

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